

Key Polynomials

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January 12, 2013

1 Introduction.

The notion of key polynomials (and the related notion of augmented valuations) was first introduced in 1936 by S. MacLane in the case of discrete rank 1 valuations (see [5], [6] and [7]).

Let $K \rightarrow L$ be a field extension and ν a valuation of K . The original motivation for introducing key polynomials was the problem of describing all the extensions μ of ν to L .

Take a valuation μ of L extending the valuation ν . In the case when ν is discrete of rank 1 and L is a simple algebraic extension of K MacLane introduced the notions of key polynomials for μ and augmented valuations and proved that μ is obtained as a limit of a family of augmented valuations on the polynomial ring $K[x]$ ([6], p. 377, Theorem 8.1).

Objects very closely related to key polynomials, called approximate roots, appeared in 1973 in the work of Abhyankar and Moh ([1] and [2]), and independently in the Thèse d'Etat of Monique Lejeune-Jalabert [4]. See also [8] for another version of the theory of approximate roots in regular two-dimensional local rings. More recently, the notion of key polynomials appeared in the work of Spivakovsky and Teissier on the local uniformization theorem (a local version of resolution of singularities) in arbitrary characteristic.

The relation between key polynomials and resolution of singularities (in the special case of singularities of plane curves) can be briefly described as follows. Let $(C, 0)$ be a germ of an irreducible plane curve defined by a polynomial $f \in k[x, y]$. Assume that f is of the form $f(x, y) = x^d + a_{d-1}(y)x^{d-1} + \dots + a_0(y)$, that is, $f(x, y) = P(x)$ with P a monic polynomial with coefficients in $K = k(y)$. If we call L the fraction field of the local ring $\mathcal{O}_{C,0}$, L is the simple extension of K defined by adjoining the variable x , satisfying the polynomial relation P . Finding a resolution of singularities of the germ $(C, 0)$ is closely related to finding valuations $\{\mu_1, \dots, \mu_r\}$ of L which extend the y -adic valuation ν of K . Precisely, resolution of singularities of $(C, 0)$ amounts to finding a regular semi-local birational ring extension \mathcal{O}' of $\mathcal{O}_{C,0}$. The localizations of \mathcal{O}' at its various maximal ideals are exactly the valuation rings $R_{\mu_1}, \dots, R_{\mu_r}$. In the case when the germ is analytically irreducible, there exists a unique extension μ of ν to L ; the family A of augmented valuations that determines μ is finite and the associated family of key polynomials $(\phi_i)_{0 \leq i \leq g}$ is the family of approximate roots of f in the sense of Abhyankar – Moh – Lejeune-Jalabert. The germs of curves defined by these polynomials have maximal contact with the curve C .

In a series of papers [9]–[13] M. Vaquié generalized MacLane's notion of key polynomials to the case of arbitrary valuations ν (that is, valuations which are not necessarily discrete of rank

1). The main definitions from these papers are reproduced below in §2. In the present paper, we will refer to key polynomials in the sense of Vaquié as **Vaquié key polynomials**. The MacLane–Vaquié approach is axiomatic in the sense that key polynomials are defined in terms of their abstract valuation-theoretic properties rather than by explicit formulae.

In the paper [3], published in the Journal of Algebra in 2007—the same year as [11]—F.J. Herrera Govantes, M.A. Olalla Acosta and M. Spivakovsky develop their own notion of key polynomials for extensions $(K, \nu) \rightarrow (L, \mu)$ of valued fields, where ν is of archimedean rank 1 (not necessarily discrete) and give an explicit description of the limit key polynomials (which can be viewed as “generalized Artin–Schreier polynomials”). These authors give a definition of a **complete system** $\{Q_i\}_{i \in \Lambda}$ of key polynomials, indexed by a well ordered set Λ of order type at most \mathbb{N} if $\text{char } \frac{R_\nu}{m_\nu} = 0$ and at most $\omega \times \omega$ if $\text{char } \frac{R_\nu}{m_\nu} > 0$. In the present paper, we will refer to key polynomials in the sense of Herrera–Olalla–Spivakovsky as **HOS key polynomials**. The definition of a complete system of HOS key polynomials is recalled below (Definition 3.1). In [3] HOS key polynomials $\{Q_i\}_{i \in \Lambda}$ are constructed by transfinite recursion in i , along with **truncations** ν_i of μ . Each Q_i is described by an explicit formula in terms of the previously defined key polynomials $\{Q_j\}_{j < i}$. Some of the main definitions and results from [3] are reproduced in §3.

Although it seemed very plausible that the two notions of key polynomials are equivalent or at least closely related, to this author’s knowledge no precise results to this effect exist in the literature. Our purpose in this paper is to clarify the relationship between the two notions of key polynomials already developed in [3] and [9]–[13]. Our main results, stated and proved in §4, can be summarized as follows:

Let $(K, \nu) \rightarrow (L, \mu)$ be an extension of valued fields with $\text{rk } \nu = 1$. Let $\{Q_i\}_{i \in \Lambda}$ be a complete system of HOS key polynomials and $\{\nu_i\}_{i \in \Lambda}$ the corresponding truncations of μ . Then:

1. For each $i \in \Lambda$ the polynomial Q_i is a Vaquié key polynomial for the truncation ν_{i_0} , where $i_0 = i - 1$ in the case when i has an immediate predecessor, and $i_0 < i$ is a suitably chosen element of Λ , described in more detail in §3, if i is a limit ordinal (this is our Proposition 4.1). As a matter of notation, we write $i = i_0 +$, regardless of whether or not i is a limit ordinal.
2. The family $F = \{\nu_i\}_{i \in \Lambda}$ of valuations constructed in [3] can be extended to an admitted family for the valuation μ (this is our Theorem 4.1).

Conversely, if $F = \{\mu_i\}_{i \in \Lambda}$ is an admissible family of valuations of $K[x]$ which is admitted for μ , and $\{Q_i\}_{i \in \Lambda}$ the family of key polynomials in the sense of Vaquié associated to F , then every polynomial Q_i in the family can be written recursively in terms of the polynomials $Q_{i'}$ with $i' < i$ (this is our Proposition 4.3). If the family F contains continued subfamilies (see Definition 2.7 below), the set Λ need not be well ordered. We end §4 by explaining how replacing Λ by a suitable well ordered subset and then suitably modifying the polynomials Q_i (roughly speaking, by subtracting terms of higher value in the sense defined precisely below) results in a complete family of HOS key polynomials.

In §5 we give an example of a limit key polynomial in the case when $\text{rk } \nu = 1$, $\text{char } \frac{R_\nu}{m_\nu} > 0$ and the valuations ν and μ are centered in local noetherian rings with fields of fractions K and L , respectively.

I thank Mark Spivakovsky for his advice and Olga Kashcheyeva for the correction of the example of the last section.

I thank the referee for useful comments and suggestions, which led to a major rewriting of the paper.

Notation: We will use the notation \mathbb{N} for the set of strictly positive integers and \mathbb{N}_0 for the set of non-negative integers.

2 Vaquié key polynomials: the definitions.

Definition 2.1. Let $\nu : K^* \rightarrow \Gamma$ be a valuation. Let (R_ν, M_ν, k_ν) denote the valuation ring of ν . For $\beta \in \Gamma$, consider the following R_ν -submodules of K :

$$P_\beta = \{y \in K^* \mid \nu(y) \geq \beta\} \cup \{0\}$$

$$P_{\beta+} = \{y \in K^* \mid \nu(y) > \beta\} \cup \{0\}$$

We define

$$G_\nu = \bigoplus_{\beta \in \Gamma} \frac{P_\beta}{P_{\beta+}}$$

The k_ν -algebra G_ν is an integral domain. For any element $y \in K^*$ with $\nu(y) = \beta$, the natural image of y in $\frac{P_\beta}{P_{\beta+}} \subset G_\nu$ is a homogeneous element of G_ν of degree β , which we will denote by $\text{in}_\nu y$.

Let (K, ν) be a valued field, x an independent variable, and let μ be a valuation of $K[x]$, extending ν .

Definition 2.2. For all f and g in $K[x]$:

1. We say that f and g are μ -equivalent if $\text{in}_\mu f = \text{in}_\mu g$.
2. We say that g μ -divides f if there exists $h \in K[x]$ such as f is μ -equivalent to $h.g$.

Definition 2.3. 1. We say that a polynomial ϕ in $K[x]$ is μ -minimal if, for all f in $K[x]$ we have: ϕ μ -divides $f \Rightarrow \deg_x f \geq \deg_x \phi$.

2. We say that ϕ is μ -irreducible if for all f, g in $K[x]$ we have:
 ϕ μ -divides $f.g \Rightarrow \phi$ μ -divides f or ϕ μ -divides g .

Definition 2.4. ([10], page 3442) A polynomial ϕ in $K[x]$ is said to be a **Vaquié key polynomial** for the valuation μ of $K[x]$ if ϕ satisfies:

1. ϕ is μ -minimal.
2. ϕ is μ -irreducible.
3. ϕ is monic.

Example 2.1. Let k be a field and $k(x, y)$ an extension of k , where x and y are two elements, algebraically independent over k . Put $K = k(y)$. Let ν be the y -adic valuation on K (in particular, $\nu(y) = 1$). Define the valuation μ on $K[x]$ as follows: for all $f = \sum_{j=0}^m b_j x^j$ in $K[x]$

$$\mu(f) = \min_{0 \leq j \leq m} \left\{ \nu(b_j) + \frac{3}{2}j \right\} \text{ (in particular, we have } \frac{3}{2} = \mu(x)).$$

For any $c \in k$ the polynomial $\phi = x^2 + cy^3$ is a Vaquié key polynomial for the valuation μ .

Let μ be a valuation of $K[x]$ and ϕ a key polynomial for μ . We note that every polynomial f in $K[x]$ can be written uniquely in the form

$$f = f_m \phi^m + f_{m-1} \phi^{m-1} + \dots + f_0$$

with $\deg_x f_j < \deg_x \phi$ for all $0 \leq j \leq m$.

Let Γ' be an ordered abelian group containing the value group Γ of μ . Take an element $\gamma \in \Gamma'$ satisfying $\gamma > \mu(\phi)$.

Definition 2.5. Define the valuation μ' by $\mu'(f) = \min_{0 \leq j \leq m} \{\mu(f_j) + j\gamma\}$. We call the valuation μ' defined by the valuation μ , the key polynomial ϕ and the element γ an **augmented valuation** and we denote: $\mu' = [\mu; \mu'(\phi) = \gamma]$.

Example 2.2. Keep the notation and hypotheses of Example 2.1. Take $c = 1$, so that $\phi = x^2 + y^3$. We see that $\mu(\phi) = \min \left\{ \frac{3}{2} * 2, \nu(y^3) \right\} = 3$.

Put $\gamma = \frac{10}{3}$ and define the augmented valuation μ' on $K[x]$ as follows:

for every polynomial $f = f_m \phi^m + f_{m-1} \phi^{m-1} + \dots + f_0$ of $K[x]$,
 $\mu'(f) = \min_{0 \leq j \leq m} \{\mu(f_j) + j\gamma\}$ where $\gamma = \mu'(\phi) = \frac{10}{3} > \mu(\phi) = 3$.

Notation. In the situation of Definition 2.5, we will sometimes write $[\mu; \mu'(\phi) = \gamma]$ instead of μ' , to emphasize the dependence of μ' on ϕ and γ .

Definition 2.6. ([10], page 3463). A family $\{\mu_\alpha\}_{\alpha \in A}$ of valuations of $K[x]$, indexed by a totally ordered set A , is called a **family of augmented iterated valuations** if for all α in A , except α the smallest element of A , there exists θ in A , $\theta < \alpha$, such that the valuation μ_α is an augmented valuation of the form $\mu_\alpha = [\mu_\theta; \mu_\alpha(\phi_\alpha) = \gamma_\alpha]$, and if we have the following properties:

1. If α admits an immediate predecessor in A , θ is that predecessor, and in the case when θ is not the smallest element of A , the polynomials ϕ_α and ϕ_θ are not μ_θ -equivalent and satisfy $\deg \phi_\theta \leq \deg \phi_\alpha$;
2. if α does not have an immediate predecessor in A , for all β in A such that $\theta < \beta < \alpha$, the valuations μ_β and μ_α are equal to the augmented valuations, respectively,

$$\mu_\beta = [\mu_\theta; \mu_\beta(\phi_\beta) = \gamma_\beta]$$

and

$$\mu_\alpha = [\mu_\beta; \mu_\alpha(\phi_\alpha) = \gamma_\alpha],$$

and the polynomials ϕ_α and ϕ_β have the same degree.

Definition 2.7. ([10], page 3464) A family of augmented iterated valuations $\{\mu_\alpha\}_{\alpha \in A}$ is said to be **continued** if there exists a valuation μ on $K[x]$, an infinite subset $\Lambda = \{\gamma_\alpha \mid \alpha \in A\}$ of the group Γ not containing a maximal element, a family of polynomials $\{\phi_\alpha\}_{\alpha \in A}$ of the same degree d , each polynomial ϕ_α being a key polynomial for the valuation μ with $\mu_\alpha = [\mu; \mu_\alpha(\phi_\alpha) = \gamma_\alpha]$ for all α in A .

Remark 2.1. ([10], 3463, beginning of §1.4) Consider a continued family of augmented iterated valuations $\{\mu_\alpha\}_{\alpha \in A}$. Then all the valuations μ_α have the same value group. In what follows, we will denote this common value group by Γ_\bullet .

Definition 2.8. ([10], page 3464) A continued family of augmented iterated valuations $\{\mu_\alpha\}_{\alpha \in A}$ is said to be **exhaustive** if the set Λ satisfies:

$$\forall \alpha < \beta \in A, \forall \gamma \in \Gamma, \gamma_\alpha < \gamma < \gamma_\beta \Rightarrow \gamma \in \Lambda.$$

Consider a continued family of augmented iterated valuations $F = \{\mu_\alpha\}_{\alpha \in A}$ as above, not necessarily exhaustive. Following [10], page 3455, let Φ_\bullet denote the set of monic polynomials ϕ in $K[x]$, of degree d , such that there exist $\alpha, \beta \in A$, depending on ϕ , satisfying $\mu_\alpha(\phi) < \mu_\beta(\phi) = \mu_{\beta'}(\phi)$ for all $\beta' \in A$ with $\beta' \geq \beta$. Let $\Lambda_\bullet = \{\mu_\beta(\phi) \mid \phi \in \Phi_\bullet, \beta \in A \text{ sufficiently large}\}$. Let $Exh(A)$ denote a totally ordered index set, with a fixed order isomorphism $\gamma : Exh(A) \rightarrow \Phi_\bullet$. We will write γ_α for $\gamma(\alpha)$ (the only reason we introduce an extra index set at this point is to make the notation consistent with that of [10]). For each $\alpha \in Exh(A)$ pick and fix a polynomial $\phi_\alpha \in \Phi_\bullet$ such that $\mu_\beta(\phi_\alpha) = \gamma_\alpha$ for all $\beta \in A$ sufficiently large; by definition of Φ_\bullet there exists $\alpha_0 \in A$ such that

$$\mu_{\alpha_0}(\phi_\alpha) < \gamma_\alpha. \quad (1)$$

Let $\mu_\alpha = [\mu_{\alpha_0}; \mu_\alpha(\phi_\alpha) = \gamma_\alpha]$; the valuation μ_α does not depend on the choice of α_0 satisfying (1).

The discussion preceding Lemme 1.17 ([10], page 3455) shows that the resulting family $Exh(F) := \{\mu_\alpha\}_{\alpha \in Exh(A)}$ is a continued family of augmented iterated valuations.

Proposition 2.1. ([10], page 3455, lemme 1.17) *Consider a continued family of augmented iterated valuations $\{\mu_\alpha\}_{\alpha \in Exh(A)}$ described above. We have the following results:*

1. *All the valuations μ_α , $\alpha \in Exh(A)$ have the same value group Γ_\bullet .*
2. *For all α in $Exh(A)$, the interval $]\gamma, \gamma_\alpha] = \{\delta \in \Gamma_\bullet \mid \gamma < \delta \leq \gamma_\alpha\}$ is contained in Λ_\bullet .*

In particular, the continued family $Exh(F) = \{\mu_\alpha\}_{\alpha \in Exh(A)}$ of augmented iterated valuations is exhaustive.

Corollary 2.1. *Proposition 2.1 shows that for any continued family $F = \{\mu_\alpha\}_{\alpha \in A}$ of augmented iterated valuations we can always add new valuations to the family in order to make the resulting family $Exh(F)$ exhaustive.*

Let f and g be two polynomials of $K[x]$. We say that f A -divides g or that g is A -divisible by f , if there exists $\alpha_0 \in A$ such that f μ_α -divides g for all $\alpha \in A$ with $\alpha > \alpha_0$.

Definition 2.9. ([10], page 3465) *A polynomial ϕ of $K[x]$ is said to be a **limit key polynomial** for the family of valuations $\{\mu_\alpha\}_{\alpha \in A}$ if ϕ has the following properties:*

- *ϕ is monic.*
- *ϕ is A -minimal, that is to say any polynomial f A -divisible by ϕ is of degree greater than or equal to ϕ*
- *ϕ is A -irreducible, that is to say: for all f, g in $K[x]$, if ϕ A -divides fg , then ϕ A -divides f or ϕ A -divides g .*

Example 2.3. *For an example of a limit key polynomial, we refer the reader to [10], page 3478. Another example is given at the end of this paper. One advantage of our example over that of [10] is that it features valuations ν and μ centered in a local noetherian ring.*

Now take a family $\{\mu_\alpha\}_{\alpha \in A}$ of augmented iterated valuations.

Remark 2.2. *It can be proved that every monic polynomial ϕ satisfying $\mu_\alpha(\phi) < \mu_\beta(\phi)$ for all $\alpha < \beta$ in A , and with a minimal degree among those satisfying this inequality, is a limit key polynomial for the family ([10], page 3465, Proposition 1.21).*

We want to define a valuation μ' of $K[x]$ starting from the family of augmented iterated valuations $\{\mu_\alpha\}_{\alpha \in A}$, from a limit key polynomial ϕ for the family $\{\mu_\alpha\}_{\alpha \in A}$, and from a value λ in Γ' that satisfies $\lambda > \mu_\alpha(\phi)$ for all $\alpha \in A$.

Consider an f in $K[x]$ such that: there exists $\alpha_0 \in A$ with $\mu_{\alpha_0}(f)$ constant for all $\alpha \in A$ such that $\alpha \geq \alpha_0$. We denote

$$\mu_A(f) = \mu_{\alpha_0}(f) = \sup\{\mu_\alpha(f) \mid \alpha \in A\}.$$

Put $f = f_m\phi^m + f_{m-1}\phi^{m-1} + \dots + f_0$, then define μ' as:

$$\mu'(f) = \inf\{\mu_A(f_j) + j\lambda; 0 \leq j \leq m\}.$$

As $\deg(f_j) < \deg(\phi)$ for all $0 \leq j \leq m$ then $\mu_A(f_j)$ is well defined for all $0 \leq j \leq m$.

Definition 2.10. We call the valuation μ' defined above the **limit augmented valuation** for the family $\{\mu_\alpha\}_{\alpha \in A}$. We denote $\mu' = [(\mu_\alpha)_{\alpha \in A}; \mu'(\phi) = \gamma]$.

Definition 2.11. ([10], page 3471). A family S of augmented iterated valuations is said to be a **simple admissible family** if it has the form $S = \{\mu_i\}_{i \in I}$, where the set of indices I is the disjoint union $I = B \cup A$, with $B \subset \mathbb{N}$ and A a totally ordered set possibly empty, where the total order on the set I is defined by $i < \alpha$ for all i in B and for all α in A , and the following properties hold:

- For $i \in B$, $i \geq 2$, we have the inequality $\deg \phi_i > \deg \phi_{i-1}$.
- If $A \neq \emptyset$, then B is finite, $B = \{1, \dots, n\}$ and for $\alpha \in A$, we have $\deg \phi_\alpha = \deg \phi_n$, and the family $\{\mu_\alpha\}_{\alpha \in A}$ is an exhaustive family of augmented iterated valuations.

If the set A is empty, i.e if the set of indices is a subset I of \mathbb{N} , we say that the family $S = \{\mu_i\}_{i \in I}$ is a **simple discrete admissible family**.

Definition 2.12. ([10], page 3472). A family of valuations $S = \{\mu_i\}_{i \in I}$ is said to be **admissible** if it is a union of a finite set or a countable set of admissible simple families $S^{(t)} = \{\mu_i^{(t)}\}_{i \in I^{(t)}}$, with $1 \leq t < N$ where $N \in \mathbb{N} \cup \{+\infty\}$, and $I^{(t)} = \{1^{(t)}, \dots, n^{(t)}\} \cup A^{(t)}$, satisfying:

- All the simple admissible families $S^{(t)} = \{\mu_i^{(t)}\}_{i \in I^{(t)}}$, except possibly the last one in the case $N < +\infty$, are non-discrete simple admissible families.
- The first valuation of the family, i.e. the first valuation $\mu_1^{(1)}$ of the first simple admissible family $S^{(1)}$, is an augmented valuation of the valuation ν of the field K constructed with the key polynomial $\phi_1^{(1)}$ of degree one.
- For $t \geq 2$, the first valuation $\mu_1^{(t)}$ of the simple admissible family $S^{(t)}$ is the limit augmented valuation for the family of valuations $\{\mu_\alpha^{(t-1)}\}_{\alpha \in A^{(t-1)}}$.

Remark 2.3. The set of indices $I = \bigcup_{t=1}^N I^{(t)}$ is totally ordered by $i < j$ for all $i \in I^{(t)}$ and $j \in I^{(s)}$, if $t < s$.

Proposition 2.2. ([10], page 3472) For all the polynomials f in $K[x]$ the family $\{\mu_i(f)\}_{i \in I}$ is increasing, which means that for all $i < j$ in I , we have $\mu_i(f) \leq \mu_j(f)$.

Furthermore, if there exists $i < j$ in I such that $\mu_i(f) = \mu_j(f)$, then for all $k \geq i$, we have also the equality $\mu_i(f) = \mu_k(f)$.

Definition 2.13. ([10], page 3473) An admissible family of valuations $F = \{\mu_j\}_{j \in I}$ of $K[x]$ is said to be an **admitted family** for the valuation μ if it has the following properties:

- For all j in I and all f in $K[x]$, $\mu_j(f) \leq \mu(f)$, and we have the equality $\mu_j(f) = \mu(f)$ for f of degree strictly less than the degree of the key polynomial ϕ_j defining the valuation μ_j .
- If $S^{(t)} = \left\{ \mu_i^{(t)} \right\}_{i \in I^{(t)}}$ is a non-discrete simple admissible family contained in F ,

$$I^{(t)} = B^{(t)} \bigcup A^{(t)},$$

then for all $\theta \in A^{(t)}$ we have the equality of sets
 $\{\mu_\alpha(\phi_\alpha) \mid \alpha \in A^{(t)}, \alpha > \theta\} = \{\mu(\phi) \mid \phi \text{ monic}, \deg(\phi) = \deg(\phi_\theta), \mu_\theta(\phi) < \mu(\phi)\}.$

Remark 2.4. From the second condition of the definition above, we notice that the Vaquié limit key polynomial $\phi_{1(t+1)}$ has degree on x strictly greater than the degree of any polynomial ϕ_α , with $\alpha \in A^{(t)}$.

Definition 2.14. We say that an admitted family $F = \{\mu_j\}_{j \in I}$ for the valuation μ **converges to μ** if for all $f \in K[x]$ there exists a $j \in I$ such that $\mu(f) = \mu_j(f)$, which is equivalent to saying that for all $f \in K[x]$ we have:

$$\mu(f) = \lim_j \mu_j(f) = \text{Max} \{ \mu_j(f), j \in I \}.$$

Theorem 2.1. ([10], page 3475) For all valuation μ of $K[x]$ extending a valuation ν of K , there exists an admissible family of valuations $(\mu_i)_{i \in I}$ that converges to μ .

We recall the statement of lemme 1.4 in [10].

Lemma 2.1. ([10], page 3443) Let μ' be the valuation defined by the valuation μ , the key polynomial ϕ , and the value $\gamma \in \Gamma$, i.e $\mu' = [\mu; \mu'(\phi) = \gamma]$. Then for all f in $K[x]$ satisfying $\mu(f) = \mu'(f)$, we have:

1. there exists h in $K[x]$ with $\deg h < \deg \phi$ such that $\text{in}_{\mu'} f = \text{in}_{\mu'} h$.
2. there exists g in $K[x]$ with $\mu'(g) = \mu(g)$ such that $\text{in}_{\mu'} fg = \text{in}_{\mu'} 1$.

We recall the statement of Proposition 1.1 and Proposition 1.2 in [9].

Proposition 2.3. ([9], page 397) Let μ be a valuation of $K[x]$. Let ϕ_1 and ϕ_2 be two key polynomials for the valuation μ , and let $\gamma_1 > \mu(\phi_1)$ and $\gamma_2 > \mu(\phi_2)$ be two values of a totally ordered group containing the ordered group of μ . Then the augmented valuations $\mu_1 = [\mu; \mu_1(\phi_1) = \gamma_1]$ and $\mu_2 = [\mu; \mu_2(\phi_2) = \gamma_2]$ defined by these polynomials and these values are equal if and only if $\gamma_1 = \gamma_2$ and if the polynomials ϕ_1 and ϕ_2 have the same degree and satisfy $\mu(\phi_1 - \phi_2) \geq \gamma_1 = \gamma_2$. In this case, the polynomials ϕ_1 and ϕ_2 are μ -equivalent.

Proposition 2.4. ([9], page 398) Let $\{\mu_\alpha\}_{\alpha \in A}$ be a continued admissible family of valuations of $K[x]$ and let ψ and ψ' be two limit key polynomials for the family $\{\mu_\alpha\}_{\alpha \in A}$, then the polynomials ψ and ψ' are μ_α -equivalent for all α sufficiently large. Furthermore the limit augmented valuations $\mu_1 = [(\mu_\alpha)_{\alpha \in A}; \mu_1(\psi) = \gamma]$ and $\mu'_1 = [(\mu_\alpha)_{\alpha \in A}; \mu'_1(\psi') = \gamma']$ defined, respectively, by ψ and ψ' and the values γ and γ' are equal if and only if $\gamma = \gamma'$ and if the polynomials ψ and ψ' satisfy $\mu_A(\psi - \psi') \geq \gamma > \mu_\alpha(\psi) = \mu_\alpha(\psi')$.

3 HOS key polynomials: definitions and some basic results.

The paper [3] define a well ordered set $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ of key polynomials of a valuation μ recursively in i . As the definition of HOS key polynomials is long, we can not repeat it all, we refer the reader to the paper [3] for a detailed definition. But, in this section we will summarize the main aspects of the definition of HOS key polynomials, and also recall some basic definitions and results from [3].

As in the previous section, let (K, ν) be a valued field, Γ the value group of ν , x an independent variable, and let μ be a valuation of $K[x]$, extending ν , with values in an ordered abelian group Γ' . For an element $\beta \in \Gamma'$, let $P'_\beta = \{y \in K[x] \mid \mu(y) \geq \beta\}$. Let Γ'_1 denote the smallest non-zero isolated subgroup of Γ' . Assume that $rk \nu = 1$.

Notation. For an element $l \in \Lambda$, we will denote by $l + 1$ the immediate successor of l in Λ . The immediate predecessor of l , when it exists, will be denoted by $l - 1$. For a positive integer t , $l + t$ will denote the immediate successor of $l + (t - 1)$.

We take this opportunity to correct a misprint in the definition of a complete set of HOS key polynomials in [3]. The correct definition is:

Definition 3.1. ([3], page 1038) *A complete set of HOS key polynomials for μ is a well ordered collection*

$$\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$$

of elements of $K[x]$ such that for each $\beta \in \Gamma'$ the additive group P'_β is generated by products of the form $a \cdot \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, $a \in K$, such that $\sum_{j=1}^s \gamma_j \mu(Q_{i_j}) + \nu(a) \geq \beta$.

Note, in particular, that if \mathbf{Q} is a complete set of HOS key polynomials then their images $\text{in}_\mu Q_i \in G_\mu$ generate G_μ as a G_ν -algebra.

Remark 3.1. *The results of [3] are stated and proved for simple **algebraic** extensions*

$$K \hookrightarrow K(x)$$

of valued fields, but those cited in the present paper are equally valid (with the same proofs) for simple pure transcendental extensions.

We look for complete systems of HOS key polynomials such that the order type of the well ordered set Λ is the smallest possible (it is shown in [3] that this order type is at most \mathbb{N} is $\text{char } \frac{R_\nu}{m_\nu} = 0$ and at most $\omega \times \omega$ if $\text{char } \frac{R_\nu}{m_\nu} > 0$).

Notation. For $l \in \Lambda$, \mathbf{Q}_l will stand for $\{Q_i\}_{i < l}$ and β_l for $\mu(Q_l)$.

The paper [3] constructs, recursively in i , HOS key polynomials $\{Q_i\}_{i \in \Lambda}$ and strictly positive integers $\{\alpha_i\}_{i \in \Lambda}$ such that for each $l \in \Lambda$ all but finitely many of the α_i with $i \leq l$ are equal to 1. We will describe below the main aspects of this construction.

Take a polynomial $h = \sum_{i=0}^s d_i x^i \in K[x]$, $d_i \in K$.

Definition 3.2. ([3], page 1039) *The first Newton polygon of h with respect to ν is the convex hull $\Delta_1(h)$ of the set $\bigcup_{i=0}^s ((\nu(d_i), i) + (\Gamma_+ \oplus \mathbb{Q}_+))$ in $\Gamma \oplus \mathbb{Q}$.*

To an element $\beta_1 \in \Gamma'_+$, we associate the following valuation ν_1 of $K[x]$: for a polynomial $h = \sum_{i=0}^s d_i x^i$, we put

$$\nu_1(h) = \min \{ \nu(d_i) + i\beta_1 \mid 0 \leq i \leq s \}.$$

Consider an element $\beta_1 \in \Gamma'_+$.

Definition 3.3. ([3], page 1039) We say that β_1 **determines a side** of $\Delta_1(h)$ if the following condition holds. Let

$$S_1(h, \beta_1) = \{ i \in \{0, \dots, s\} \mid i\beta_1 + \nu(d_i) = \nu_1(h) \}.$$

We require that $\#S_1(h, \beta_1) \geq 2$.

Let $\beta_1 = \mu(x)$. Then for any $h \in K[x]$ we have

$$\nu_1(h) \leq \mu(h) \tag{2}$$

by the axioms for valuations. If equality holds in (2) for all $h \in K[x]$, we put $\Lambda = \{1\}$, $x = Q_1$ and stop. The definition of key polynomials is complete. From now on, assume that there exists a polynomial $h \in K[x]$ such that $\nu_1(h) < \mu(h)$.

Proposition 3.1. ([3], page 1039) Take a polynomial $h = \sum_{i=0}^s d_i x^i \in K[x]$ such that

$$\nu_1(h) < \mu(h).$$

Then

$$\sum_{i \in S(h, \beta_1)} \text{in}_\nu d_i \text{in}_\mu x^i = 0.$$

Corollary 3.1. ([3], page 1040) Take a polynomial $h \in K[x]$ such that $\nu_1(h) < \mu(h)$. Then β_1 determines a side of $\Delta_1(h)$.

Notation. Let X be a new variable. Take a polynomial h as above. We denote

$$\text{in}_1 h := \sum_{i \in S_1(h, \beta_1)} \text{in}_\nu d_i X^i.$$

The polynomial $\text{in}_1 h$ is quasi-homogeneous in $G_\nu[X]$, where the weight assigned to X is β_1 . Let

$$\text{in}_1 h = v \prod_{j=1}^t g_j^{\gamma_j} \tag{3}$$

be the factorization of $\text{in}_1 h$ into irreducible factors in $G_\nu[X]$. Here $v \in G_\nu$ and the g_j are monic polynomials in $G_\nu[X]$ (to be precise, we first factor $\text{in}_1 h$ over the field of fractions of G_ν and then observe that all the factors are quasi-homogeneous and therefore lie in $G_\nu[X]$).

Proposition 3.2. ([3], page 1040)

1. The element $\text{in}_\mu x$ is integral over G_ν .
2. The minimal polynomial of $\text{in}_\mu x$ over G_ν is one of the irreducible factors g_j on the right hand side of (3).

Now let g_1 be the minimal polynomial of $\text{in}_\mu x$ over G_ν . Let $\alpha_2 = \deg_X g_1$. Write $g_1 = \sum_{i=0}^{\alpha_2} \bar{b}_i X^i$, where $\bar{b}_{\alpha_2} = 1$. For each i , $0 \leq i \leq \alpha_2$, choose a representative b_i of \bar{b}_i in R_ν (that is, an element of R_ν such that $\text{in}_\nu b_i = \bar{b}_i$; in particular, we take $b_{\alpha_2} = 1$). Put $Q_2 = \sum_{i=0}^{\alpha_2} b_i x^i$.

Definition 3.4. ([3], page 1041) *The elements Q_1 and Q_2 are called, respectively, the first and second key polynomials of μ .*

Now, every element y of $K[x]$ can be written uniquely as a finite sum of the form

$$y = \sum_{\substack{0 \leq \gamma_1 < \alpha_2 \\ 0 \leq \gamma_2}} b_{\gamma_1 \gamma_2} Q_1^{\gamma_1} Q_2^{\gamma_2} \quad (4)$$

where $b_{\gamma_1 \gamma_2} \in K$ (this is proved by Euclidean division by the monic polynomial Q_2). The expression (4) is called **the second standard expansion of y** .

Now, take an ordinal number greater than or equal to 3 which has an immediate predecessor; denote this ordinal by $l+1$. If $\nu(\mathbb{N}) = 0$, assume that $l \in \mathbb{N}_0$. Assume given a set \mathbf{Q}_{l+1} of polynomials and positive integers $\alpha_{l+1} = \{\alpha_i\}_{i \leq l}$, such that $\mu(Q_i) \in \Gamma'_1$ for $i \leq l$ and all but finitely many of the α_i are equal to 1. Furthermore, we assume that for each $i \leq l$ the polynomial Q_i has an explicit expression in terms of \mathbf{Q}_i , described below.

We will use the following multi-index notation: $\bar{\gamma}_{l+1} = \{\gamma_i\}_{i \leq l}$, where all but finitely many γ_i are equal to 0, $\mathbf{Q}_{l+1}^{\bar{\gamma}_{l+1}} = \prod_{i \leq l} Q_i^{\gamma_i}$. Let $\beta_i = \mu(Q_i)$.

Definition 3.5. ([3], page 1041) *An index $i < l$ is said to be **l -essential** if there exists a positive integer t such that either $i+t = l$ or $i+t < l$ and $\alpha_{i+t} > 1$; otherwise i is called **l -inessential**.*

In other words, i is l -inessential if and only if $i + \omega \leq l$ and $\alpha_{i+t} = 1$ for all $t \in \mathbb{N}_0$.

Notation. For $i < l$, let

$$\begin{aligned} i+ &= i+1 && \text{if } i \text{ is } l\text{-essential} \\ &= i+\omega && \text{otherwise.} \end{aligned}$$

Definition 3.6. ([3], page 1041)

*A multiindex $\bar{\gamma}_{l+1}$ is said to be **standard with respect to α_{l+1}** if*

$$0 \leq \gamma_i < \alpha_{i+} \text{ for } i \leq l, \quad (5)$$

*and if i is l -inessential then the set $\{j < i+ \mid j+ = i+ \text{ and } \gamma_j \neq 0\}$ has cardinality at most one. An **l -standard monomial in \mathbf{Q}_{l+1}** (resp. an **l -standard monomial in $\text{in}_\mu \mathbf{Q}_{l+1}$**) is a product of the form $c_{\bar{\gamma}_{l+1}} \mathbf{Q}_{l+1}^{\bar{\gamma}_{l+1}}$, (resp. $c_{\bar{\gamma}_{l+1}} \text{in}_\mu \mathbf{Q}_{l+1}^{\bar{\gamma}_{l+1}}$) where $c_{\bar{\gamma}_{l+1}} \in K$ (resp. $c_{\bar{\gamma}_{l+1}} \in G_\nu$) and the multiindex $\bar{\gamma}_{l+1}$ is standard with respect to α_{l+1} .*

Remark 3.2. ([3], page 1042) *In the case when i is l -essential, the condition (5) amounts to saying that $0 \leq \gamma_i < \alpha_{i+1}$.*

Definition 3.7. ([3], page 1042) *An **l -standard expansion not involving Q_l** is a finite sum S of l -standard monomials, not involving Q_l , having the following property. Write $S = \sum_{\beta} S_{\beta}$, where β ranges over a certain finite subset of Γ'_+ and*

$$S_{\beta} = \sum_j d_{\beta j} \quad (6)$$

is a sum of standard monomials $d_{\beta j}$ of value β . We require that

$$\sum_j in_\mu d_{\beta j} \neq 0 \quad (7)$$

for each β appearing in (6).

Proposition 3.3. ([3], page 1042) Let i be an ordinal and t a positive integer. Assume that $i + t + 1 \leq l$, so that the key polynomials \mathbf{Q}_{i+t+1} are defined, and that $\alpha_i = \dots = \alpha_{i+t} = 1$. Then any $(i+t)$ -standard expansion does not involve any Q_q with $i \leq q < i+t$. In particular, an i -standard expansion not involving Q_i is the same thing as an $(i+t)$ -standard expansion, not involving Q_{i+t} .

We will frequently use this fact in the sequel without mentioning it explicitly.

Definition 3.8. ([3], page 1042) For an element $g \in K[x]$, an expression of the form $g = \sum_{j=0}^s c_j Q_l^j$, where each c_j is an l -standard expansion not involving Q_l , will be called an **l -standard expansion** of g .

Definition 3.9. ([3], page 1042) Let $\sum_{\bar{\gamma}} \bar{c}_{\bar{\gamma}} in_\mu \mathbf{Q}_{l+1}^{\bar{\gamma}}$ be an l -standard expansion, where $\bar{c}_{\bar{\gamma}} \in G_\nu$. A **lifting** of $\sum_{\bar{\gamma}} \bar{c}_{\bar{\gamma}} in_\mu \mathbf{Q}_{l+1}^{\bar{\gamma}}$ to $K[x]$ is an l -standard expansion $\sum_{\bar{\gamma}} c_{\bar{\gamma}} \mathbf{Q}_{l+1}^{\bar{\gamma}}$, where $c_{\bar{\gamma}}$ is a representative of $\bar{c}_{\bar{\gamma}}$ in K .

Definition 3.10. ([3], page 1042) Assume that $\text{char } k_\nu = p > 0$. An l -standard expansion $\sum_j c_j Q_l^j$, where each c_j is an l -standard expansion not involving Q_l , is said to be **weakly affine** if $c_j = 0$ whenever $j > 0$ and j is not of the form p^e for some $e \in \mathbb{N}_0$.

Assume, inductively, that for each ordinal $i \leq l$, every element h of $K[x]$ admits an i -standard expansion. Furthermore, assume that for each $i \leq l$, the i -th polynomial Q_i admits an i_0 -standard expansion, with $i = i_0 +$, having the following additional properties:

If i has an immediate predecessor $i - 1$ in Λ (such is always the case if $\text{char } k_\nu = 0$), the $(i - 1)$ -st standard expansion of Q_i has the form

$$Q_i = Q_{i-1}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} \left(\sum_{\bar{\gamma}_{i-1}} c_{ji\bar{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\bar{\gamma}_{i-1}} \right) Q_{i-1}^j, \quad (8)$$

where:

1. each $c_{ji\bar{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\bar{\gamma}_{i-1}}$ is an $(i - 1)$ -standard monomial, not involving Q_{i-1}
2. the quantity $\nu(c_{ji\bar{\gamma}_{i-1}}) + j\beta_{i-1} + \sum_{q < i-1} \gamma_q \beta_q$ is constant for all the monomials

$$\left(c_{ji\bar{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\bar{\gamma}_{i-1}} \right) Q_{i-1}^j$$

appearing on the right hand side of (8)

3. the equation

$$in_\mu Q_{i-1}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} \left(\sum_{\bar{\gamma}_{i-1}} in_\nu c_{ji\bar{\gamma}_{i-1}} in_\mu \mathbf{Q}_{i-1}^{\bar{\gamma}_{i-1}} \right) in_\mu Q_{i-1}^j = 0 \quad (9)$$

is the minimal algebraic relation satisfied by $in_\mu Q_{i-1}$ over $G_\nu[in_\mu \mathbf{Q}_{i-1}]$.

Finally, if $\text{char } k_\nu = p > 0$ and i does not have an immediate predecessor in Λ then there exist an i -inessential index i_0 and a strictly positive integer e_i such that $i = i_0 +$ and

$$Q_i = c_{0i_0} + \sum_{j=0}^{e_i} c_{p^j i_0} Q_{i_0}^{p^j} \quad (10)$$

is a weakly affine monic i_0 -standard expansion of degree $\alpha_i = p^{e_i}$ in Q_{i_0} , where each $c_{q i_0}$ is an i_0 -standard expansion not involving Q_{i_0} . Moreover, there exists a positive element $\bar{\beta}_i \in \Gamma'$ such that

$$\begin{aligned} \bar{\beta}_i &> \beta_q && \text{for all } q < i, \\ \beta_i &\geq p^{e_i} \bar{\beta}_i && \text{and} \\ p^j \bar{\beta}_i + \nu(c_{p^j i_0}) &= p^{e_i} \bar{\beta}_i && \text{for } 0 \leq j \leq e_i. \end{aligned}$$

Definition 3.11. *The set \mathbf{Q}_{l+1} is called an l -th set of **HOS Key Polynomial**. By a set of **HOS key polynomials** we will mean a set \mathbf{Q} of polynomials for which there exists an ordinal l such that \mathbf{Q} is an l -th set of key polynomials. We will loosely refer to elements of this set as **HOS key polynomials**.*

If $i \in \mathbb{N}_0$, one can prove by induction that the i -standard expansion is unique. If $\text{char } k_\nu > 0$ and $h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$ is an i -standard expansion of h (where $h \in K[x]$), then the elements $d_{ji} \in K[x]$ are uniquely determined by h (strictly speaking, this does not mean that the i -standard expansion is unique: for example, if i is a limit ordinal, d_{ji} admits an i_0 -standard expansion for each $i_0 < i$ such that $i = i_0 +$, but there may be countably many choices of i_0 for which such an i_0 -standard expansion is an i_0 -standard expansion, not involving Q_{i_0} in the sense of Definition 3.7).

Definition 3.12. *For each ordinal $i \leq l$ we define a valuation ν_i of L as follows. Given an i -standard expansion $h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$, put*

$$\nu_i(h) = \min_{0 \leq j \leq s_i} \{j\beta_i + \mu(d_{ji})\}. \quad (11)$$

The valuation ν_i will be called the i -truncation of ν .

Note that even though in the case when $\text{char } k_\nu > 0$ the standard expansions of the elements d_{ji} are not, in general, unique, the elements $d_{ji} \in K[x]$ themselves are unique by Euclidean division, so ν_i is well defined. That ν_i is, in fact, a valuation, rather than a pseudo-valuation, follows from the definition of standard expansion, particularly, from (7). We always have

$$\nu_i(h) \leq \mu(h).$$

The paper [3] constructs, starting with a set \mathbf{Q}_{l+1} of HOS key polynomials, a polynomial Q_{l+1} such that \mathbf{Q}_{l+2} forms an $(l+1)$ -st set of key polynomials (this means, in other words, that Q_{l+1} which has the form (8) and satisfies properties 1–3). If $\alpha_{l+1} = 1$ it may happen that the construction of [3] produces a $(l + \omega + 1)$ -st set of HOS key polynomials, that is, an infinite sequence of polynomials $\{Q_{l+t}\}_{t \in \mathbb{N}}$ and a polynomial $Q_{l+\omega}$ (referred to as the limit HOS key polynomial) which has the form (10) and satisfies the properties listed right after equation (10).

We will finish our construction here, for more details of the construction, we refer the reader to the paper [3]. We also note that the paper [3] section 8 page 1068 proves that we can always construct a **complete** family of HOS key polynomials.

We end this section giving more definitions and results of [3] that we will use in the rest of our paper.

Proposition 3.4. ([3], page 1044)

1. The polynomial Q_i is monic in x ; we have

$$\deg_x Q_i = \prod_{j \leq i} \alpha_j.$$

2. Let z be an i -standard expansion, not involving Q_i . Then

$$\deg_x z < \deg_x Q_i.$$

We recall a result from the statement of Corollary 25 in [3]:

Proposition 3.5. ([3], page 1045, Corollary 25) We have

$$\begin{aligned} \beta_i &> \alpha_i \beta_{i-1} && \text{if } (i-1) \text{ exists} \\ \beta_i &> p^{e_i} \overline{\beta_i} && \text{otherwise.} \end{aligned}$$

Proposition 3.6. ([3], page 1051) Consider an ordinal $l \in \Lambda$. Let y be a polynomial in $K[x]$ of degree strictly less than $\deg_x(Q_{l+1}) = \prod_{i=1}^{l+1} \alpha_i$. Then $\mu(y) = \nu_l(y)$.

Definition 3.13. Let $h = \sum_{j=0}^s d_{ji} Q_i^j$ be an i -standard expansion, let $\overline{Q_i}$ be a variable, and let $\beta_i = \mu(Q_i)$. We define

$$S_i(h, \beta_i) := \{j \in \{0, \dots, s\} \mid j\beta_i + \mu(d_{ji}) = \nu_i(h)\}$$

$$in_i(h) := \sum_{j \in S_i(h, \beta_i)} in_\mu d_{ji} \overline{Q_i}^j$$

We define $\delta_i(h) := \deg_{\overline{Q_i}} in_i(h)$.

We recall a result from the statement of Proposition 37 in [3]:

Proposition 3.7. ([3], page 1044) We have $\alpha_{i+1} \delta_{i+1}(h) \leq \delta_i(h)$.

4 The main results: comparison of Vaquié and HOS key polynomials.

Let the notation be as in the previous sections, with $rk \nu = 1$.

Proposition 4.1. We assume that the family of HOS key polynomials $\mathbf{Q}_i = \{Q_i\}_{i \in \Lambda}$ is already defined ([3], part 3). Let i be an ordinal and let $i_0 = i - 1$ if i admits an immediate predecessor and i_0 as in (10) otherwise.

Then Q_i is a Vaquié key polynomial for the valuation ν_{i_0} .

Proof. :

1. Q_i is ν_{i_0} -minimal:

Suppose that Q_i ν_{i_0} -divides f , then there exists $h \in K[x]$ such that

$$\nu_{i_0}(f - hQ_i) > \nu_{i_0}(f) = \nu_{i_0}(hQ_i).$$

On the other hand we have: $\mu(hQ_i) = \mu(h) + \beta_i > \nu_{i_0}(h) + \alpha_i \beta_{i_0} = \nu_{i_0}(hQ_i) = \nu_{i_0}(f)$ where the strict inequality holds by Proposition 3.5 .

Then we have the inequality

$$\mu(f) \geq \inf\{\mu(f - hQ_i), \mu(hQ_i)\} \geq \inf\{\nu_{i_0}(f - hQ_i), \mu(hQ_i)\} > \nu_{i_0}(f).$$

Then $\deg_x f \geq \deg_x Q_i$ because otherwise by Proposition 3.6 we would have

$$\mu(f) = \nu_{i_0}(f).$$

2. Q_i is ν_{i_0} -irreducible:

Suppose that Q_i ν_{i_0} -divides $f.g$, as above, we find

$$\mu(f.g) > \nu_{i_0}(f.g)$$

Now, either $\mu(f) > \nu_{i_0}(f)$ and $\deg(f) \geq \deg(Q_i)$, or $\mu(g) > \nu_{i_0}(g)$ and $\deg(g) \geq \deg(Q_i)$. Suppose that $\mu(f) > \nu_{i_0}(f)$ and let $f = qQ_i + r$ be the Euclidean division of f by Q_i with $q \neq 0$ because $\deg(f) \geq \deg(Q_i)$, and $\mu(r) = \nu_{i_0}(r)$ because $\deg(r) < \deg(Q_i)$.

Hence

$$\begin{aligned} \nu_{i_0}(r) = \mu(r) &\geq \inf(\mu(f), \mu(qQ_i)) > \inf(\nu_{i_0}(f), \nu_{i_0}(qQ_i)) \\ &\Rightarrow \nu_{i_0}(r) > \nu_{i_0}(f) = \nu_{i_0}(qQ_i) \end{aligned}$$

and Q_i ν_{i_0} -divides f .

3. Q_i is monic by definition. □

Remark 4.1. Let i and i_0 be as in Proposition 4.1. As HOS key polynomials are also Vaquié key polynomials for ν_{i_0} , and as $\mu(d_{ji})$ on the right side of (11) in Definition 3.12 is equal to $\nu_{i_0}(d_{ji})$ by Proposition 3.6, the i -truncation ν_i is also an augmented valuation defined by the valuation ν_{i_0} and the key polynomial Q_i .

Corollary 4.1. Assume that there exists an ordinal i and a strictly positive integer t such that $\deg_x Q_i = \deg_x Q_{i+t}$. Then the polynomial Q_{i+t} is also a Vaquié key polynomial for the valuation ν_{i_0} .

Proof. The polynomial Q_{i+t} can be written as $Q_{i+t} = Q_i + z_t$ where z_t is an i -standard expansion not involving Q_i with $\nu_i(Q_i) = \nu_i(z_t)$. We have $\nu_{i_0}(z_t) = \nu_i(z_t)$ by Proposition 3.6. Now, $\nu_i(Q_i) > \nu_{i_0}(Q_i)$, so $\nu_{i_0}(z_t) > \nu_{i_0}(Q_i) = \nu_{i_0}(Q_{i+t})$. Hence Q_i and Q_{i+t} are ν_{i_0} -equivalent. □

Let $\{Q_i\}_{i \in \Lambda}$ be a family of HOS key polynomials constructed in [3], and $\{\nu_i\}_{i \in \Lambda}$ the corresponding family of valuations.

Take an ordinal $i+1 \in \Lambda$ which admits an immediate predecessor i , and such that $\deg_x Q_i = \deg_x Q_{i+1}$. Let Δ_i be a totally ordered set such that there exists a bijection Φ between $[\beta_i; \beta_{i+1}] \subset \nu_{i+1}(K[x])$ and Δ_i .

Lemma 4.1. *For all $\beta \in]\beta_i; \beta_{i+1}[$, there exists a polynomial $Q_\delta \in K[x]$ with $\delta = \Phi(\beta)$ in Δ_i which satisfies :*

$$Q_\delta = Q_i + z_\delta$$

with z_δ an i -standard expansion not involving Q_i ,

$$\nu_i(Q_\delta) = \nu_i(Q_i) = \nu_i(z_\delta)$$

and

$$\nu_{i+1}(Q_\delta) = \beta_\delta > \beta_i.$$

Proof. Take any $\beta \in]\beta_i; \beta_{i+1}[$, put $\delta = \Phi(\beta) \in \Delta_i$.

As $\beta \in \nu_{i+1}(K[x])$ then there exist $h_\beta \in K[x]$ such that $\nu_{i+1}(h_\beta) = \beta$. As $\nu_{i+1}(h_\beta) = \beta < \beta_{i+1}$ then $\mu(h_\beta) = \nu_{i+1}(h_\beta)$ and by Lemma 2.1 there exists an $h_\delta \in K[x]$, such that

$$\deg_x(h_\delta) < \deg_x(Q_{i+1}) = \deg_x(Q_i)$$

and $\mu(h_\delta) = \mu(h_\beta) = \beta$.

Put $Q_\delta = Q_{i+1} + h_\delta$. We notice that $\nu_{i+1}(Q_\delta) = \nu_{i+1}(h_\delta) = \mu(h_\delta) < \beta_{i+1} = \nu_{i+1}(Q_{i+1})$.

As $Q_{i+1} = Q_i + z_i$ with $\deg_x(z_i) < \deg_x(Q_i)$, and $\nu_i(Q_i) = \mu(z_i)$,

then $Q_\delta = Q_i + (z_i + h_\delta)$, therefore $\nu_i(Q_\delta) = \nu_i(Q_i) = \beta_i$ because $\nu_i(h_\delta) = \beta > \beta_i$.

Hence, $Q_\delta = Q_i + z_\delta$, with $z_\delta = h_\delta + z_i$, with $\deg_x(z_\delta) < \deg_x(Q_i)$ and

$$\nu_i(Q_\delta) = \nu_i(Q_i) < \nu_{i+1}(Q_\delta) = \mu(Q_\delta).$$

Put $\beta_\delta = \mu(Q_\delta) = \beta$, then we have

$$Q_\delta = Q_i + z_\delta$$

with z_δ an i -standard expansion not involving Q_i ,

$$\nu_i(Q_\delta) = \nu_i(Q_i) = \nu_i(z_\delta)$$

and $\beta_\delta > \beta_i$.

□

Fix $\delta \in \Delta_i$ and take the polynomial Q_δ defined above. Put $\nu_\delta = [\nu_i; \nu_\delta(Q_\delta) = \beta_\delta]$.

By Lemma 4.1 and proposition 4.1, the polynomials $\{Q_i\}_{i \in \Delta_i}$ are Vaquié key polynomials for the valuation ν_i .

Proposition 4.2. *The family $(\nu_\delta)_{\delta \in \Delta_i}$ associated to the key polynomials $(Q_\delta)_{\delta \in \Delta_i}$ is an augmented iterated family of valuations.*

Proof. :

Take $\delta_1 < \delta_2 \in \Delta_i$, we have $Q_{\delta_1} = Q_i + z_{\delta_1}$, and $Q_{\delta_2} = Q_i + z_{\delta_2}$. Therefore $Q_{\delta_2} = Q_{\delta_1} + (z_{\delta_2} - z_{\delta_1})$, from this relation we can see that the polynomial Q_{δ_2} is a key polynomial for the valuation ν_{δ_1} and that the valuation ν_{δ_2} is the augmented valuation constructed by the valuation ν_{δ_1} and the key polynomial Q_{δ_2} .

Now as we have, $\deg_x(Q_\delta) = \deg_x(Q_i)$, $\forall \delta \in \Delta_i$, we still have to prove that Q_{δ_1} and Q_{δ_2} are not ν_{δ_1} -equivalent. We have, $z_{\delta_1} = z_i + h_{\delta_1}$ and $z_{\delta_2} = z_i + h_{\delta_2}$ with $\mu(h_{\delta_1}) = \Phi^{-1}(\delta_1)$ and $\mu(h_{\delta_2}) = \Phi^{-1}(\delta_2)$, and $\Phi^{-1}(\delta_1) < \Phi^{-1}(\delta_2)$. Hence $\nu_{\delta_1}(Q_{\delta_2} - Q_{\delta_1}) = \nu_{\delta_1}(z_{\delta_2} - z_{\delta_1}) = \nu_{\delta_1}(h_{\delta_2} - h_{\delta_1}) = \mu(h_{\delta_2} - h_{\delta_1}) = \Phi^{-1}(\delta_1) = \nu_{\delta_1}(Q_{\delta_1}) = \nu_{\delta_1}(Q_{\delta_2})$.

This completes the proof.

□

The result of Proposition 4.2 is closely related to the result of Proposition 2.1 (due to Vaquié). Although the latter is sufficient for the purposes of this paper, we have kept Proposition 4.2 because we feel that it clarifies the nature of $Exh(F)$ and the relation between Vaquié and HOS key polynomials.

In fact, by Proposition 2.1 we can extend any continued family F of augmented iterated valuations to an exhaustive family $Exh(F)$.

We note, using Proposition 4.1, that the polynomial $Q_{l+\omega}$ defined in the seventh part of [3] is a Vaquié limit key polynomial for the family $\{Q_{l+t}\}_{t \in \mathbb{N}_0}$.

Theorem 4.1. *The family $F = \{\nu_i\}_{i \in \Lambda}$ constructed in [3] can be extended to an admitted family $Exh(F)$ for the valuation μ .*

Proof. We will proceed by the order of construction of the Q_i .

We take $Q_{1(1)} = x$ and $\nu_{1(1)}$. If $\nu_{1(1)}(h) = \mu(h) \forall h \in K[x]$ we have finished, we take $Exh(F) = \{\nu_{1(1)}\}$.

If not, consider the polynomial Q_2 . If there exists an integer t_0 such that $\deg_x(Q_{2+t}) = \deg_x Q_2$ for all $t \leq t_0$, Q_{2+t_0+1} is defined and $\deg_x(Q_{2+t_0+1}) > \deg_x Q_2$, then by Corollary 4.1 the polynomial Q_{2+t_0} is a Vaquié key polynomial for the valuation $\nu_{1(1)}$. We put $Q_{2(1)} = Q_{2+t_0}$ and $\nu_{2(1)} = [\nu_{1(1)}; \nu_{2(1)}(Q_{2(1)}) = \mu(Q_{2(1)})]$. We use the same procedure to construct the valuations $\nu_{3(1)}, \nu_{4(1)}, \dots$

If we have $\alpha_i > 1$ for infinitely many values of i , we set $Exh(F) = \{\nu_i^{(1)}\}_{i \in I^{(1)}}$, with $I = I^{(1)} = \{1, \dots, n, \dots\}$. Take any element $h \in K[x]$. From Proposition 3.7 we have

$$\delta_{i+1}(h) < \delta_i(h) \quad \text{for } i, i+1 \in I^{(1)}. \quad (12)$$

As the set $I^{(1)}$ is infinite and the strict inequality (12) cannot occur infinitely many times, we have $\delta_i(h) = 0$ for some i . Then $\nu_i(h)$ does not involve \overline{Q}_i , hence $\nu_i(h) = \mu(h)$ and we have finished.

If not, i.e. if there exists a certain l such that $\alpha_l = \alpha_{l+1} = \alpha_{l+2} = \dots = 1$, we set

$$I^{(1)} = B^{(1)} \cup Exh(A^{(1)})$$

with $B^{(1)} = \{1^{(1)}, \dots, l^{(1)}\}$ and $A^{(1)} = \{l^{(1)} + 1, l^{(1)} + 2, \dots\}$ where $l^{(1)}$ is the minimal l satisfying $\alpha_{l+t} = 1$ for all $t \in \mathbb{N}_0$.

If $\forall h$ in $K[x]$, there exists $i \in A^{(1)}$ such as $\nu_i(h) = \mu(h)$ we have finished.

If not, we know the existence of a limit key polynomial $Q_{l+\omega}$ and a valuation limit $\nu_{l+\omega}$, which satisfies $\nu_{l+\omega}(f) \leq \mu(f)$ for all $f \in K[x]$. We denote: $\nu_{l+\omega} = \nu_1^{(2)}$ and we repeat the procedure. In this way, we construct recursively an admissible family of augmented iterated valuations which is admitted for the valuation μ . \square

Conversely, given an admissible family of valuations F of $K[x]$ which is admitted for the valuation μ , we want to see how to obtain from the family of Vaquié key polynomials associated to F , a family of HOS key polynomials.

We will first prove an analogue of Lemma 2.1 when Q_i is a limit key polynomial.

Lemma 4.2. *Let $C = \{\mu_\alpha\}_{\alpha \in A}$ be a continued family of augmented iterated valuations, and $\{\phi_\alpha\}_{\alpha \in A}$ the set of the key polynomials associated to C .*

Let μ be the valuation defined by the family C , a limit key polynomial ϕ and a value $\gamma = \mu(\phi)$. Then for all f in $K[x]$ for which there exists $\alpha_0 \in A$ such that for all $\alpha \geq \alpha_0$ $\mu_\alpha(f) = \mu(f)$, we have:

1. there exists h in $K[x]$ with $\deg h < \deg \phi$ such that $\text{in}_\mu f = \text{in}_\mu h$.
2. there exists g in $K[x]$ with $\deg g < \deg \phi$ such that $\text{in}_\mu f g = \text{in}_\mu 1$.

Proof. 1. Let $f = q\phi + r$ be the Euclidean division of f by ϕ .

As $\deg_x r < \deg_x \phi$, there exists $\alpha_1 \in A$ such that for all $\alpha \geq \alpha_1$ in A , we have $\mu_\alpha(r) = \mu(r)$.

Take $\alpha_2 = \max\{\alpha_0, \alpha_1\}$ then for all $\alpha \geq \alpha_2$, we have $\mu_\alpha(q\phi) \geq \mu_\alpha(f)$.

Indeed, suppose that there exists $\beta \in A$, $\beta \geq \alpha_2$ and $\mu_\beta(q\phi) < \mu_\beta(f)$. Then $\mu_\beta(r) = \mu_\beta(q\phi)$ and for all $\alpha \geq \beta \geq \alpha_2$ we have $\mu_\alpha(f) \geq \mu_\beta(f) > \mu_\beta(r) = \mu_\alpha(r)$. Hence for all $\alpha \geq \beta$ we have $\text{in}_{\mu_\alpha} r = \text{in}_{\mu_\alpha} q\phi$ and ϕ A -divides r , which contradicts the fact that $\deg_x r < \deg_x \phi$ because ϕ is μ_A -minimal.

Hence, for all $\alpha \geq \alpha_2$ we have $\mu(q\phi) > \mu_\alpha(q\phi) \geq \mu_\alpha(f) = \mu(f)$, therefore f is μ -equivalent to r .

2. As the limit key polynomial ϕ is an irreducible polynomial of $K[x]$ and ϕ does not divide f , therefore there exist two polynomials g and h of $K[x]$, with $\deg_x g < \deg_x \phi$, such that $fg + h\phi = 1$, hence $\text{in}_\mu f g = \text{in}_\mu 1$.

□

Let $F = \{\mu_i\}_{i \in I}$ be an admissible family of valuations of $K[x]$ which is admitted for μ , and let $\{Q_i\}_{i \in I}$ be the family of key polynomials in the sense of Vaquié associated to F . For all $i \in I$ put $\beta_i = \mu_i(Q_i)$.

Write $F = \bigcup_t S^{(t)} = \bigcup_t \left\{ \mu_i^{(t)} \right\}_{i \in I^{(t)}}$, with $1 \leq t < N$ where $N \in \mathbb{N} \cup \{+\infty\}$, and

$$I^{(t)} = \left\{ 1^{(t)}, \dots, n^{(t)} \right\} \bigcup A^{(t)}.$$

Theorem 4.2. *There exist well ordered sets I' and J , $I' \subset I$, $I' \subset J$, and a polynomial Q'_i for each $i \in J$, having the following properties:*

1. I' is cofinal in both I and J .
2. $1^{(t)} \in I'$ for all t , $1 \leq t < N$.
3. the set $(Q'_i)_{i \in J}$ is a J -set of HOS key polynomials.
4. for any two consecutive elements $i_0, i_1 \in I'$, there is at most one $j \in J$ such that

$$i_0 < j < i_1. \tag{13}$$

If there exists $j \in J$ satisfying (13) then $\deg Q'_j = \deg Q_{i_1}$.

5. For every $i \in I'$ there exists $i_0 \in I'$ with $i = i_0 +$ such that the key polynomial Q'_i satisfies $\mu_{i_0}(Q_i - Q'_i) > \mu(Q_i) = \mu(Q'_i)$.
6. If the family F converges to μ then the set $(Q'_i)_{i \in J}$ of HOS key polynomials is complete.

Proof. We start the proof of Theorem 4.2 with an auxiliary Proposition which gives explicit formulae expressing each Vaquié key polynomial Q_i appearing in F in terms of polynomials $Q_{i'}$ with $i' < i$.

Proposition 4.3. *For every i in I there exists an $i_0 \in I$, $i_0 < i$ such that the polynomial Q_i can be written as:*

$$Q_i = Q_{i_0}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} \left(\sum_{\gamma_{i_0}} c_{ji\gamma_{i_0}} Q_{i_0}^{\gamma_{i_0}} \right) Q_{i_0}^j \quad (14)$$

where:

1. Each $c_{ji\gamma_{i_0}} Q_{i_0}^{\gamma_{i_0}}$ is an i_0 -standard monomial not involving Q_{i_0} .
2. We have $j\beta_{i_0} + \mu(c_{ji\gamma_{i_0}} Q_{i_0}^{\gamma_{i_0}}) \geq \alpha_i\beta_{i_0}$ for all the monomials $(c_{ji\gamma_{i_0}} Q_{i_0}^{\gamma_{i_0}}) Q_{i_0}^j$ appearing in (14).
3. We have $\beta_i > \alpha_i\beta_{i_0}$.
4. Q_i is a polynomial of minimal degree among those satisfying $\nu_{i_0}(Q_i) < \mu(Q_i)$.

Proof. Write $F = \bigcup_t S^{(t)} = \bigcup_t \left\{ \mu_i^{(t)} \right\}_{i \in I^{(t)}}$, with $1 \leq t < N$ where $N \in \mathbb{N} \cup \{+\infty\}$, and

$$I^{(t)} = \left\{ 1^{(t)}, \dots, n^{(t)} \right\} \bigcup A^{(t)}.$$

We know that the first valuation, ν_1^1 is constructed in the same way by M.Vaquié and by HOS, with the key polynomial $Q_1^1 = x$ or $Q_1^1 = x - a$ with $a \in K$.

Now we have three cases:

Case 1 $i = i^{(t)} \in I^{(t)}$ with $i \in \{2^{(t)}, \dots, n^{(t)}\}$.

Put $i_0 = (i-1)^{(t)}$, and let $Q_i = f_m Q_{i_0}^m + f_{m-1} Q_{i_0}^{m-1} + \dots + f_0$.

In the case when Q_{i_0} is a key polynomial (in the sense of Vaquié), Vaquié proves in [10] (Théorème 1.11, page 9) that $f_m = 1$ and that $\mu_{i_0}(Q_i) = m\beta_{i_0}$ where $\beta_{i_0} = \mu_{i_0}(Q_{i_0}) = \mu(Q_{i_0})$.

Now if i_0 is a limit ordinal and Q_{i_0} is a limit key polynomial (in the sense of Vaquié), this means that we are in the case $t \neq 1$ and $(i-2)^{(t)}$ does not exist. We will first prove the Proposition in this case.

We will denote $A = A^{(t-1)}$.

As $\deg_x f_m < \deg_x Q_{i_0}$, there exists $\alpha_0 \in A$ such that for all $\alpha \in A$, $\alpha \geq \alpha_0$, we have $\mu_{i_0}(f_m) = \mu_\alpha(f_m)$. By lemma 4.2 there exists g in $K[x]$ with $\deg_x g < \deg_x Q_{i_0}$ such that $in_{\mu_{i_0}} f_m g = in_{\mu_{i_0}} 1$.

As $\deg_x g < \deg_x Q_{i_0}$, there exists $\alpha_1 \in A$ such that for all $\alpha \in A$, $\alpha \geq \alpha_1$, we have $\mu_{i_0}(g) = \mu_\alpha(g)$.

As for all j , $0 \leq j \leq m-1$, $\deg_x f_j < \deg_x Q_{i_0}$, then for all j , $0 \leq j \leq m-1$, there exists $\alpha_{1,j} \in A$ such that for all $\alpha \in A$, $\alpha \geq \alpha_{1,j}$, we have $\mu_{i_0}(f_j) = \mu_\alpha(f_j)$.

Take $\alpha_2 = \max \{\alpha_1, \alpha_{1,0}, \dots, \alpha_{1,j}, \dots, \alpha_{1,m-1}\}$, then for all $\alpha \geq \alpha_2$, and for all j , $0 \leq j \leq m-1$, $\mu_{i_0}(f_j g) = \mu_\alpha(f_j g)$, therefore by Lemma 4.2, for all j , $0 \leq j \leq m-1$, there exists h_j in $K[x]$ with $\deg_x h_j < \deg_x Q_{i_0}$ such that $in_{\mu_{i_0}} f_j g = in_{\mu_{i_0}} h_j$.

Put $\varphi = Q_{i_0}^m + h_{m-1} Q_{i_0}^{m-1} + \dots + h_0 Q_{i_0}$, we have $in_{\mu_{i_0}} \varphi = in_{\mu_{i_0}} g Q_i$ so that Q_i μ_{i_0} -divides φ . Therefore $\deg_x \varphi \geq Q_i$, hence $\deg_x f_m = 0$, and as Q_i is monic in x because Q_i is a Vaquié key polynomial, then $f_m = 1$.

Now we have $\mu_{i_0}(Q_i) \leq m\beta_{i_0}$. If we have $\mu_{i_0}(Q_i) < m\beta_{i_0}$ (where $\beta_{i_0} = \mu(Q_{i_0}) = \mu_{i_0}(Q_{i_0})$) then we will have $in_{\mu_{i_0}} Q_i = in_{\mu_{i_0}} (Q_i - Q_{i_0}^m)$ which contradicts the fact that Q_i is μ_{i_0} -minimal. Therefore $\mu_{i_0}(Q_i) = m\beta_{i_0}$.

We have $Q_i = Q_{i_0}^m + f_{m-1}Q_{i_0}^{m-1} + \dots + f_0$, with $\mu_{i_0}(Q_i) = m\beta_{i_0} = m\mu_{i_0}(Q_{i_0})$.

As for all j , $0 \leq j \leq m-1$, we have $\deg_x f_j < \deg_x Q_{i_0}$ hence we can write $f_j = \sum_{\gamma_{i_0}} c_{ji\gamma_{i_0}} \mathbf{Q}_{i_0}^{\gamma_{i_0}}$

where $c_{ji\gamma_{i_0}} \mathbf{Q}_{i_0}^{\gamma_{i_0}}$ is an i_0 -standard monomial not involving Q_{i_0} .

We have $j\beta_{i_0} + \mu(\sum_{\gamma_{i_0}} c_{ji\gamma_{i_0}} \mathbf{Q}_{i_0}^{\gamma_{i_0}}) = j\beta_{i_0} + \mu(f_j) \geq \mu_{i_0}(Q_i) = m\beta_{i_0}$.

Finally, by definition of μ_i , we have $\mu(Q_i) > \mu_{i_0}(Q_i)$, and we have proved that $\mu_{i_0}(Q_i) = m\beta_{i_0}$, then $\mu_i(Q_i) > m\beta_{i_0}$.

Case 2 $i = i^{(t)} \in I^{(t)}$ such that $i \in A^{(t)}$.

Pick any α in $A^{(t)}$ such that $\alpha < i$; note that if i is the first element of $A^{(t)}$ we take $\alpha = n^{(t)}$ the final element of the discrete set of the simple admissible family $S^{(t)}$. Take $i_0 = \alpha$.

We know that $\deg_x(Q_i) = \deg_x(Q_{i_0})$, therefore we can write

$$Q_i = Q_{i_0} + z_{i_0}$$

where $z_{i_0} \in K[x]$ with $\deg_x z_{i_0} < \deg_x Q_{i_0}$.

We have $\mu_{i_0}(Q_i) = \min\{\mu_{i_0}(Q_{i_0}), \mu_{i_0}(z_{i_0})\}$,

if $\mu_{i_0}(Q_{i_0}) > \mu_{i_0}(Q_i)$, then Q_i is μ_{i_0} -equivalent to z_{i_0} , which contradicts the fact that Q_i is μ_{i_0} -minimal.

Hence $\mu_{i_0}(Q_{i_0}) = \mu_{i_0}(Q_i)$.

Moreover, we have $\mu_i(Q_i) > \mu_{i_0}(Q_i) = \min\{\mu_{i_0}(Q_{i_0}), \mu_{i_0}(z_{i_0})\} = \min\{\mu_i(Q_{i_0}), \mu_i(z_{i_0})\}$, hence

$$\mu_i(Q_i) > \mu_i(Q_{i_0}) = \mu_i(z_{i_0}).$$

Case 3 $i = 1^{(t)}$, and $t \neq 1$, this is the case when Q_i is a limit key polynomial for the continued family of valuations $(\mu_i)_{i \in A^{(t-1)}}$. As the rank of the group Γ is equal to 1, the subset $\Lambda^{t-1} := \{\mu(Q_\alpha), \alpha \in A^{(t-1)}\}$ does not admit a largest element but an upper bound in Γ . By [9] (Theorem 3.5, page 33) there exists an integer m , such that for α sufficiently large in $A^{(t-1)}$, we have:

$$Q_i = Q_\alpha^m + f_{m-1}Q_\alpha^{m-1} + \dots + f_0 \quad (15)$$

with $\mu_\alpha(Q_i) = m\beta_\alpha = m\mu_\alpha(Q_\alpha)$.

By construction of the family F , Q_i satisfies $\mu(Q_i) > \mu_\alpha(Q_i)$ for all $\alpha < i$. We have $\mu_i(Q_i) = \mu(Q_i)$ by definition of μ_i .

Then $\mu_i(Q_i) > \mu_\alpha(Q_i) = m\beta_\alpha$ for α sufficiently large in $A^{(t-1)}$.

Therefore pick α sufficiently large in $A^{(t-1)}$ and take $i_0 = \alpha$.

By the choice of i_0 and by the first condition of Definition 2.13, Q_i has minimal degree among all the polynomials satisfying $\nu_{i_0}(Q_i) < \mu(Q_i)$.

□

Definition 4.1. A *partial collection of HOS key polynomials associated to F* is a pair of sets (I', J) and a collection of polynomials $\{Q'_i\}_{i \in J}$ having the following properties:

1. $I' \subset I$.
2. $J \supset I'$
3. I' is cofinal in J

4. $\{Q'_i\}_{i \in J}$ is a J -th set of HOS key polynomials
5. for any two consecutive elements $i_0, i_1 \in I'$, there is at most one element $j \in J$ satisfying (13). If such a j exists, we have $\deg Q'_j = \deg Q_{i_1}$.
6. For each $i \in I'$ there exists $i_0 \in I'$ with $i = i_0 +$ such that the key polynomial Q'_i satisfies $\mu_{i_0}(Q_i - Q'_i) > \mu(Q_i) = \mu(Q'_i)$.

Next, we introduce the following partial ordering on the set of all the partial collections of HOS key polynomials, associated to F .

Definition 4.2. Let (I', J) , with $\{Q'_i\}_{i \in J}$ the corresponding J -th set of HOS key polynomials and (I'', J') , with $\{Q''_j\}_{j \in J'}$ the corresponding J' -th set of HOS key polynomials, be two partial collections of HOS key polynomials, associated to F . We say that $(I', J) \preceq (I'', J')$ if there is an inclusion $I' \subset I''$, and an inclusion $J \subset J'$, such that $Q'_i = Q''_i$ for all $i \in J$.

Lemma 4.3. The set of all the partial collections of HOS key polynomials associated to F is not empty.

Proof. Let $I' = \{1^{(1)}\}$, put $Q'_{1^{(1)}} = Q_{1^{(1)}}$ and let $J = \{1^{(1)}\}$. □

The partially ordered set of all the partial collections of HOS key polynomials, associated to F , satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element.

Therefore, to prove Theorem 4.2, it remains to prove the following statement: if (I', J) is a partial collection of HOS key polynomials, associated to F , such that I' is not cofinal in I then there exists a partial collection (I'', J') of HOS key polynomials, associated to F , such that $(I', J) \prec (I'', J')$.

Let (I', J) be a partial collection of HOS key polynomials associated to F , such that I' is not cofinal in I .

To prove the above statement, we will define a partial collection (I'', J') of HOS key polynomials, associated to F , such that $(I', J) \prec (I'', J')$.

As I' is not cofinal in I , there exists $\alpha \in I$ such that $\alpha > I'$. As I' is cofinal in J we have $\alpha > J$.

We have two cases:

1. J admits a maximal element i_0 .
2. J does not admit a maximal element, but there exists a subset $E \subset I$ such that $E = I^{(1)} \cup \dots \cup I^{(t_0-1)} \cup I^{(t_0)}$ with $1 \leq t_0 < N$ and $I^{(t_0)} = \{1^{(t_0)}, \dots, n^{(t_0)}\} \cup A^{(t_0)}$ with $A^{(t_0)}$ not empty, $I' \subset E$ and I' is cofinal in E .

We want to define a set J' such that $J \subsetneq J'$.

If J admits a maximal element i_0 , as J is cofinal in I' and $I' \subset I$, we have $i_0 \in I$. Hence there exists t , $1 \leq t < N$ such that $i_0 \in I^{(t)} = \{1^{(t)}, \dots, n^{(t)}\} \cup A^{(t)}$.

If $i_0 = s^{(t)}$ with $s^{(t)} < n^{(t)}$, put $i = (s+1)^{(t)}$.

If $i_0 = n^{(t)}$ or $i_0 \in A^{(t)}$, then choose any $\alpha \in A^{(t)}$ such that $\alpha > i_0$ and put $i = \alpha$.

If J does not have a maximal element, take E as above, and put $i = 1^{(t_0+1)}$.

We will define the polynomial Q'_i .

Definition 4.3. Let t , $2 \leq t < N$. For an element i , $i \in I$, we say that i is a limit ordinal for the family F if $i = 1^{(t)}$. Otherwise we say that i is a simple ordinal for F .

Assume that i is a simple ordinal for F .

Definition 4.4. If $i \in \{1^{(t)}, \dots, n^{(t)}\}$ with $1 \leq t < N$, write Q_i as in (14).

For $j \in \{0, \dots, \alpha_i - 1\}$, let a_{ji_0} denote the sum of all the monomials $c_{ji\gamma_{i_0}} Q_{i_0}^{\gamma_{i_0}}$ of value $\mu_{i_0}(Q_i) - j\beta_{i_0} = \alpha_i\beta_{i_0} - j\beta_{i_0}$ (if the set of such monomials is empty, we put $a_{ji_0} = 0$). Put $P_i = Q_{i_0}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} a_{ji_0} Q_{i_0}^j$. If $i \in A^{(t)}$ with $1 \leq t < N$ we put $P_i = Q_i$.

Proposition 4.4. P_i is a Vaquié key polynomial for the valuation μ_{i_0} .

Proof. Indeed, P_i is monic, and P_i and Q_i are μ_{i_0} -equivalent by definition. \square

We will now define the polynomial P_i in the case when i is a limit ordinal.

Let $i = 1^{(t_0+1)}$, Q_i is a limit key polynomial in the sense of Vaquié for the continued family of valuations $\{\mu_{i'}\}_{i' \in A^{(t_0)}}$.

As the rank of the group Γ is equal to 1, the subset $\Lambda^{(t_0)} := \{\mu(Q_\alpha), \alpha \in A^{(t_0)}\}$ does not admit a maximal element but an upper bound $\overline{\beta_{t_0}}$ in Γ .

By the proof of Proposition 55 ([3] page 1063), if $f = \sum_{j=0}^s a_{ji'} Q_{i'}^j \in K[x]$ with $i' \in J$, satisfies $\mu_{i''}(f) < \mu(f)$ for all $i'' \in J$, then there exists an $i_1 \in J$ and an integer $0 < j \leq s$, $j = p^{e_0}$ for e_0 a strictly positive integer, such that for all $i' > i_1 \in J$ the polynomial $f' := b_{p^{e_0}i_1} Q_{i_1}^{p^{e_0}} + \sum_{j=0}^{p^{e_0}-1} b_{ji_1} Q_{i_1}^j$ (where b_{ji_1} is an i_1 -standard expansion not involving Q_{i_1} for all $0 \leq j < p^{e_0}$) satisfies $\mu_{i''}(f') < \mu(f')$ for all $i'' \in J'$. The integer p^{e_0} is the minimal degree in $Q_{i'}$ with $i' \in J$, of a polynomial $f \in K[x]$ satisfying $\mu_{i''}(f) < \mu(f)$ for all $i'' \in J$.

By [9] (Theorem 3.5, page 33) there exists $i_0 \in J$ such that for all $i' > i_0 \in J$, we can write $Q_i = Q_{i'}^m + \sum_{j=0}^{m-1} d_{ji'} Q_{i'}^j$ with $d_{ji'} \in K[x]$, $\deg_x d_{ji'} < \deg_x Q_{i'}$, and $\mu_{i'}(Q_i) = m\beta_{i'} = \mu(d_{0i'})$.

By the first condition of Definition 2.13 we know that Q_i is the polynomial with the minimal degree among those which satisfy $\mu_{i'}(Q_i) < \mu(Q_i)$ for all $i' \in J'$. Hence $m = p^{e_0}$.

Write $Q_i = Q_{i_0}^m + \sum_{j=0}^{m-1} a_{ji_0} Q_{i_0}^j$ with a_{ji_0} and i_0 -standard expansion not involving Q_{i_0} , then also by the proof of Proposition 55 in [3] there exists $i_1 > i_0$ and a polynomial $P_i = Q_{i_1}^{p^{e_0}} + a_{0i_1} + \sum_{j=1}^{p^{e_0}-1} a'_{ji_1} Q_{i_1}^j$ where $a'_{ji_1} = a_{ji_1}$ or $a'_{ji_1} = 0$ and P_i is a weakly affine i_1 -standard expansion with $\mu_{i_1}(P_i) = p^{e_0}\beta_{i_1} = \mu(a_{0i_1})$ and for all $0 < j \leq p^{e_0-1}$ we have $\mu(a'_{ji_1}) + j\overline{\beta} = p^{e_0}\overline{\beta}$.

As well, P_i satisfies $\mu(P_i) > \mu_{i'}(P_i)$ for all $i' \in J$. Hence P_i is a limit Vaquié key polynomial for the family $\{\mu_\alpha\}_{\alpha \in A^{(t_0)}}$.

For simplicity, we will replace i_1 by i_0 in the above definition of P_i .

Now we will define the valuation μ'_i .

Definition 4.5. Put $\beta'_i = \mu(P_i)$. If i is a simple ordinal, we define the valuation $\mu'_i := [\mu_{i_0}; \mu(P_i) = \beta'_i]$. If i is a limit ordinal $i = 1^{(t_0+1)}$, we define the valuation $\mu'_i := [\{\mu_\alpha\}_{\alpha \in A^{(t_0)}}; \mu(P_i) = \beta'_i]$.

As P_i and μ'_i are well defined, we will study the valuation μ'_i .

Remark 4.2. Put $h = Q_i - P_i$. We have $\deg_x(h) < \deg_x Q_i = \deg_x P_i$, therefore $\mu_{i_0}(h) = \mu(h)$.

Furthermore, if

$$\mu(h) \geq \beta'_i = \beta_i, \quad (16)$$

then by Proposition 2.3 and Proposition 2.4 we have $\mu'_i = \mu_i$.

Proposition 4.5. If $\mu_i(P_i) = \beta'_i$ then we have:

1. $\mu(h) \geq \beta'_i$.
2. $\beta_i \geq \beta'_i$.

Proof. 1. We have $Q_i = P_i + h$, with $\mu_{i_0}(h) = \mu(h)$ then

$$\beta_i = \mu(Q_i) \geq \min\{\mu(P_i), \mu(h)\} = \min\{\beta'_i, \mu(h)\}. \quad (17)$$

On the other hand, by definition of μ_i we have

$$\beta'_i = \mu_i(P_i) = \min\{\beta_i, \mu(h)\}. \quad (18)$$

Suppose that $\mu(h) < \beta'_i$, then from (17) $\beta_i \geq \mu(h)$ then from (18) $\beta'_i = \mu(h)$ which is impossible. Therefore we have $\mu(h) \geq \beta'_i$.

2. Now 2 follows from (17).

□

Now we can construct the partial collection of HOS key polynomials (I'', J') .

If $\mu_i = \mu$, we have $\mu_i(P_i) = \mu(P_i) = \beta'_i$ and by Proposition 4.5 we have $\mu(h) \geq \beta'_i$ and $\beta_i \geq \beta'_i$.

If $\mu(h) > \beta'_i$, or if $\mu(h) = \beta'_i$ and $\beta_i = \beta'_i$ then the condition (16) is satisfied and by Remark 4.2 we have $\mu'_i = \mu_i = \mu$.

Let l be a new index. We put $I'' = I' \cup \{l\}$ and $J' = J \cup \{l\}$, $Q'_l = P_i$. The set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials. (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

If $\mu(h) = \beta'_i$ and $\beta_i > \beta'_i$, then we have $Q_i = P_i + h$ with $\deg_x h < \deg_x P_i$ and $\beta_i > \beta'_i = \mu(h)$. We define two new indices i_1, l such that $J < i_1 < l$. We put $Q'_{i_1} = P_i$ and $Q'_l = Q_i$. We put $I'' = I' \cup \{l\}$ and $J' = J \cup \{i_1, l\}$; the set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J)$.

From now on assume that $\mu_i \neq \mu$.

If $i \in I^{(t)}$ such that $i = s^{(t)}$ with $s^{(t)} < n^{(t)}$, then $i + 1$ exists in I and $i + 1 = (i + 1)^{(t)}$ and the polynomial f with the minimal degree that satisfies $\mu(f) > \mu_i(f)$ has degree $\deg_{Q_i} Q_{i+1} > 1 = \deg_{Q_i} P_i$, hence the polynomial P_i satisfies $\mu_i(P_i) = \mu(P_i)$ and by Proposition 4.5 we have $\mu(h) \geq \beta'_i$ and $\beta_i \geq \beta'_i$.

If $\mu(h) > \beta'_i$, or if $\mu(h) = \beta'_i$ and $\beta_i = \beta'_i$ then the condition (16) is satisfied and by Remark 4.2 we have $\mu'_i = \mu_i$.

In this case, we put $I'' = I' \cup \{i\}$ and $J' = J \cup \{i\}$, $Q'_i = P_i$. The set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J)$.

If $\mu(h) = \beta'_i$ and $\beta_i > \beta'_i$, then we have $Q_i = P_i + h$ with $\deg_x h < \deg_x P_i$ and $\beta_i > \beta'_i = \mu(h)$. We define two new indices i_1, l such that $J < i_1 < l$. We put $Q'_{i_1} = P_i$ and $Q'_l = Q_i$. We put $I'' = I' \cup \{l\}$ and $J' = J \cup \{i_1, l\}$; the set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J)$.

If $i = n^{(t)}$ we have two cases:

Case 1: P_i satisfies the condition (16), we have $\mu'_i = \mu_i$. We choose a cofinal subset $D^{(t)}$ in $A^{(t)}$ which is of order type \mathbb{N} .

We put $I'' = I' \cup \{i\} \cup \{D^{(t)}\}$ and $J' = J \cup \{i\} \cup \{D^{(t)}\}$, $Q'_i = P_i$, and we take the set $\{Q'_{i'}\}_{i' \in J''}$ with $Q'_{i'} = Q_{i'}$ if $i' \in \{D^{(t)}\}$. The set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials and (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

Case 2: P_i does not satisfy the condition (16), then either $\mu(P_i) > \mu(Q_i)$ or $\mu(P_i) < \mu(Q_i)$.

Suppose that $\mu(P_i) > \mu(Q_i)$. As the family $\{\mu_\alpha\}_{\alpha \in A}$ is exhaustive, there exists a polynomial Q_α with $\alpha \in A$ such that $\mu(P_i) = \mu(Q_\alpha)$. In this case we put $Q'_i = Q_\alpha$ which is also a Vaquié key polynomial for the valuation $\mu'_{i_0} = \mu_{i-1}$.

We put :

$$A'^{(t)} = \{\alpha' \in A^{(t)} \mid \alpha' > \alpha\},$$

We choose a cofinal subset $D^{(t)}$ from $A'^{(t)}$ of order type \mathbb{N} . We put $I'' = I' \cup \{\alpha\} \cup D^{(t)}$ and $J' = J \cup \{\alpha\} \cup D^{(t)}$,

and we take the set $\{Q'_{i'}\}_{i' \in J'}$ with $Q'_{i'} = Q_{i'}$ if $i' \in \{D^{(t)}\}$. The set $\{Q'_{i'}\}_{i' \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

Now assume that $\mu(P_i) < \mu(Q_i)$. Then we have $Q_i = P_i + h$ with $\deg_x h < \deg_x P_i$ and $\beta_i > \beta'_i = \mu(h)$.

We define two new indices i_1, l such that $J < i_1 < l$. We put $Q'_{i_1} = P_i$ and $Q'_l = Q_i$. We choose a cofinal subset $D^{(t)}$ from $A^{(t)}$ of order type \mathbb{N} . We put $I'' = I' \cup \{l\} \cup \{D^{(t)}\}$ and $J' = J \cup \{i_1, l\} \cup \{D^{(t)}\}$ the set $\{Q'_j\}_{j \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

The only case that left is the case when $i = \alpha$ with $\alpha \in A^{(t)}$. In this case we have $P_i = Q_\alpha$ and $\mu'_i = \mu_\alpha$.

We choose a cofinal subset $D^{(t)}$ from $A^{(t)}$ of order type \mathbb{N} such that $D^{(t)} > \alpha$. We put $I'' = I' \cup \{i\} \cup \{D^{(t)}\}$ and $J' = J \cup \{i\} \cup \{D^{(t)}\}$ and we take the set $\{Q'_{i'}\}_{i' \in J'}$ with $Q'_{i'} = Q_{i'}$ if $i' \in \{D^{(t)}\}$. The set $\{Q'_{i'}\}_{i' \in J'}$ is a J' -th set of HOS key polynomials. Then (I'', J') is a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

We have constructed a partial collection of HOS key polynomials associated to F with $(I', J) \prec (I'', J')$.

By Zorn's Lemma, the partially ordered set of all the partial collections of HOS key polynomials, associated to F contains a maximal element, hence there exists a partial collection (I^e, J^e) of HOS key polynomials associated to F which is maximal for the partial ordering of the set of all the partial collections of HOS key polynomials associated to F . Let $\{Q'_i\}_{i \in J^e}$ be the J^e -th set of HOS key polynomials associated to (I^e, J^e) .

From the proof above, I^e is cofinal in I .

Now suppose that the family F converges to μ . then for all $f \in K[x]$, there exists $i \in I^{(e)}$ such that $\mu_i^e(f) = \mu(f)$. As I^e is cofinal in I , then for all $f \in K[x]$, there exists $i \in I^{(e)}$ such that $\mu_i^e(f) = \mu(f)$. And as $I^e \subset J^e$ therefore the set $\{Q'_j\}_{j \in J^e}$ is a complete set of HOS key polynomials for μ . □

5 Example.

In this section, we want to give an example of a limit key polynomial, such that both valuations ν and μ are centered in local noetherian rings, one of which dominates the other.

We start by giving some definitions and some properties of key polynomials and augmented valuations.

Let x be a variable and k a field. Let ν be a valuation of $k[x]$, Γ an ordered group containing $\nu(K[x])$ as a sub-group, ϕ a key polynomial for ν , and $\gamma \in \Gamma$ such that $\gamma > \nu(\phi)$.

Let $\mu = [\nu; \mu(\phi) = \gamma]$. For all $f = \sum_{j=0}^s a_j \phi^j \in K[x]$, with $a_j \in k[x]$ and $\deg_x a_j < \deg_x \phi$ for $0 \leq j \leq s$, we define $D_\phi(f) = \max \{j \in \{0, \dots, s\} \mid \mu(f) = \nu(a_j) + j\gamma\}$.

If $d = D_\phi(f)$ then by definition we have $\text{in}_\mu f = \text{in}_\mu \sum_{j=0}^d a_j \phi^j$.

We notice that the integer $D_\phi(f)$ depends only on the image $\text{in}_\mu f$ in the graded algebra G_μ , therefore if f and f' are μ -equivalent, then $D_\phi(f) = D_\phi(f')$.

We have also $D_\phi(f.g) = D_\phi(f) + D_\phi(g)$.

Lemma 5.1. *Let $f = \sum_{j=0}^s a_j \phi^j$, with $a_j \in k[x]$ and $\deg_x a_j < \deg_x \phi$ for $0 \leq j \leq s-1$ and $\deg_x a_s = 0$ then:*

$$\mu(a_s \phi^s) = \mu(f) \Rightarrow f \text{ is } \mu\text{-minimal}.$$

Proof. Let $g \in k[x]$ such that f μ -divides g .

Then there exists $h \in k[x]$ such that $\text{in}_\mu g = \text{in}_\mu h \text{in}_\mu f$, therefore $D_\phi g \geq D_\phi f$,

and for all $g \in k[x]$ we have $\deg_x g \geq D_\phi g \cdot \deg_x \phi$.

On the other hand from the hypothesis we have $\deg_x f = D_\phi f \cdot \deg_x \phi$.

Finally, we find $\deg_x g \geq D_\phi g \cdot \deg_x \phi \geq D_\phi f \cdot \deg_x \phi = \deg_x f$. □

Let k be a field of characteristic $p > 2$.

Consider the pure transcendental field extension $k(z)$ of k with the z -adic valuation ν_0 (with $\nu_0(z) = 1$).

Consider the field $K = k(y, z)$ where y is a pure transcendental element over $k(z)$. We define the valuation ν of K which extends ν_0 in the following way:

Put :

$$\begin{aligned} Q_{y,1} &= y & \gamma_{y,1} &= \frac{1}{2} \\ Q_{y,2} &= y^2 + z & \gamma_{y,2} &= p - \frac{1}{4} \\ Q_{y,3} &= Q_{y,2}^2 + z^{2p-1}y & \gamma_{y,3} &= 2p - \frac{1}{8p} \\ Q_{y,4} &= Q_{y,3}^{2p} - z^{4p^2-p}Q_{y,2} & \gamma_{y,4} &= 4p^2 - \frac{1}{16} \\ Q_{y,5} &= Q_{y,4}^2 + z^{6p^2}Q_{y,3}^p & \gamma_{y,5} &= 8p^2 - \frac{1}{32p} \end{aligned}$$

For $j > 2$ put:

$$\begin{aligned} Q_{y,2j} &= Q_{y,2j-1}^{2p} + z^{2^{2j-2}p^j - 2^{2j-4}p^{j-1}}Q_{y,2j-2} & \gamma_{y,2j} &= 2^{2j-2}p^j - \frac{1}{2^{2j}} \\ Q_{y,2j+1} &= Q_{y,2j}^2 + z^{3(2)^{2j-3}p^j}Q_{y,2j-1}^p & \gamma_{y,2j+1} &= 2^{2j-1}p^j - \frac{1}{2^{2j+1}p} \end{aligned}$$

Now we define recursively for all $j \geq 1$ the augmented valuation $\nu_j = [\nu_{j-1}; \nu_j(Q_{y,j}) = \gamma_j]$. In fact, the construction of ν_1 is obvious. And we notice that every polynomial $Q_{y,j}$ is a key polynomial for the valuation ν_{j-1} .

Indeed, each polynomial $Q_{y,j}$ is monic, and by lemma 5.1, for all $j \geq 1$, $Q_{y,j}$ is ν_{j-1} -minimal.

Now to prove that $Q_{y,2j}$ is ν_{2j-1} -irreducible, it is sufficient to prove that the image of the monomial $z^{2^{2j-2}p^j - 2^{2j-4}p^{j-1}}Q_{y,2j-2}$, in the graded algebra $G_{\nu_{2j-1}}$ is neither a square 2 nor a p -th power, i.e it is sufficient to prove that $\nu_{2j-1}(z^{2^{2j-2}p^j - 2^{2j-4}p^{j-1}}Q_{y,2j-2})$ is not divisible by either 2 or p in the group $\mathbb{Z} + \nu(Q_{y,1})\mathbb{Z} + \dots + \nu(Q_{y,2j-1})\mathbb{Z}$. As $in_{\nu_{2j-1}}Q_{y,2j-1}$ is transcendental over $in_{\nu_{2j-1}}\mathbf{Q}_{y,2j-1}$ in $G_{\nu_{2j-1}}$, it is sufficient to prove that $\nu_{2j-1}(z^{2^{2j-2}p^j - 2^{2j-4}p^{j-1}}Q_{y,2j-2})$ is not divisible by either 2 or p in the group $\mathbb{Z} + \nu(Q_{y,1})\mathbb{Z} + \dots + \nu(Q_{y,2j-2})\mathbb{Z}$. Now $\nu_{2j-1}(z^{2^{2j-2}p^j - 2^{2j-4}p^{j-1}}Q_{y,2j-2}) = 2^{2j-2}p^j - \frac{1}{2^{2j-2}} = 2p(2^{2j-3}p^{j-1} - \frac{1}{2^{2j-1}p})$. But neither $2(2^{2j-3}p^{j-1} - \frac{1}{2^{2j-1}p})$ nor $p(2^{2j-3}p^{j-1} - \frac{1}{2^{2j-1}p})$ can be in $\mathbb{Z} + \nu(Q_{y,1})\mathbb{Z} + \dots + \nu(Q_{y,2j-2})\mathbb{Z}$. Hence $Q_{y,2j}$ is ν_{2j-1} -irreducible.

And with the same method we prove that $Q_{y,2j+1}$ is ν_{2j} -irreducible.

We notice that for all $f \in k(z)[y]$, there exists $j \in \mathbb{N}$ such that $\forall i > j$, $\nu_i(f) = \nu_j(f)$. Therefore we can define the valuation ν by:

$$\forall f \in k(z)[y], \nu(f) = \max \{ \nu_j(f) \mid j \in \mathbb{N} \}$$

Now, consider the field $K(x)$ with x a pure transcendental element over K .

We define the valuation μ of $K(x)$ which extends the valuation ν in the following way:

We will first define $h_i(y, z) \in K$. Let h_i be defined by:

$$\begin{aligned}
h_1 &= \frac{Q_{y,3}^2}{z^{4p-1}} & \nu(h_1) &= 2\nu(Q_{y,3}) - \nu(z^{4p-1}) = 4p - \frac{1}{4p} - 4p + 1 = 1 - \frac{1}{4p} \\
h_i &= \frac{Q_{y,2i+1}^2}{z^{2^{2i}p^i-1}} & \nu(h_i) &= 2\nu(Q_{y,2i+1}) - \nu(z^{2^{2i}p^i-1}) = 2^{2i}p^i - \frac{1}{2^{2i}p} - 2^{2i}p^i + 1 = 1 - \frac{1}{2^{2i}p}
\end{aligned}$$

Now put : $\mu(x) = 1 - \frac{1}{4p}$. We have $\mu(x) = \mu(h_1)$, and the value of the polynomial $x - h_1$ is not determined by the values of x and h_1 . Put

$$Q_{x,1} = x - h_1 \quad \text{and} \quad \mu(Q_{x,1}) = 1 - \frac{1}{2^4p} > \mu(x).$$

we have $\mu(Q_{x,1}) = \mu(h_2)$, and the value of the polynomial $x - h_1 - h_2 = Q_{x,1} - h_2$ is not determined by the values of $Q_{x,1}$ and h_2 . Put

$$Q_{x,2} = x - h_1 - h_2 \quad \text{and} \quad \mu(Q_{x,2}) = 1 - \frac{1}{2^6p} > \mu_1(Q_{x,2}) = \mu(Q_{x,1})$$

where μ_1 is the i -truncation associated to the key polynomial $Q_{x,1}$.

We have $\mu(Q_{x,2}) = \mu(h_3)$, and the value of the polynomial $x - h_1 - h_2 - h_3 = Q_{x,2} - h_3$ is not determined by the values of $Q_{x,2}$ and h_3 . Put

$$Q_{x,3} = x - h_1 - h_2 - h_3 \quad \text{and} \quad \mu(Q_{x,3}) = 1 - \frac{1}{2^8p} > \mu_2(Q_{x,3}) = \mu(Q_{x,2})$$

where μ_2 is the i -truncation associated to the key polynomial $Q_{x,2}$.

We can construct by induction, an infinite family of key polynomials $\{Q_{x,i}\}_{i \in \mathbb{N}}$, with an infinite family of i -truncations $\{\mu_i\}_{i \in \mathbb{N}}$ associated to $\{Q_{x,i}\}_{i \in \mathbb{N}}$.

$$Q_{x,i} = x - h_1 - h_2 - \dots - h_i \quad \text{and} \quad \mu(Q_{x,i}) = 1 - \frac{1}{2^{2i+2}p}.$$

To simplify the notation we will denote $Q_i := Q_{x,i}$, and $\beta_i = \mu(Q_i) = \mu_i(Q_i)$.

For each $i \in \mathbb{N}$, we have $Q_{x,i} = Q_{x,i-1} - h_i$ and $\beta_i = \mu_i(Q_i) > \mu_{i-1}(Q_i) = \mu_{i-1}(Q_{i-1}) = \beta_{i-1}$.

The sequence $\{\beta_i\}_{i \in \mathbb{N}}$ is strictly increasing and bounded; it does not contain a maximal element. We have $\lim_{i \rightarrow \infty} \beta_i = \bar{\beta} = 1$.

Now take the polynomial $f = x^p - y^2 - z$. We want to prove that f is a limit key polynomial, that is, that f is the polynomial of smallest degree which satisfies $\mu(f) > \mu_i(f)$ for all $i \in \mathbb{N}$.

Replacing x by $Q_{i-1} = x - h_1 - h_2 - \dots - h_i$ in f we find :

$$f = Q_{i-1}^p + h_1^p + h_2^p + \dots + h_i^p - y^2 - z.$$

We notice that:

$$-y^2 - z + h_1^p = -Q_{y,2} + \frac{Q_{y,3}^{2p}}{z^{4p^2-p}} = \frac{Q_{y,4}}{z^{4p^2-p}}$$

and

$$-y^2 - z + h_1^p + h_2^p = \frac{Q_{y,4}}{z^{4p^2-p}} + \frac{Q_{y,5}^{2p}}{z^{16p^3-p}} = \frac{Q_{y,6}}{z^{16p^3-p}}.$$

It is not hard to prove that for every $i \in \mathbb{N}$ we have

$$-y^2 - z + h_1^p + h_2^p + \dots + h_{i-1}^p = \frac{Q_{y,2i}}{z^{2^{2i-2}p^i-p}}.$$

Therefore for all $i \in \mathbb{N}$ we have:

$$f = Q_{i-1}^p + \frac{Q_{y,2i}}{z^{2^{2i-2}p^i-p}}$$

with $\nu(\frac{Q_{y,2i}}{z^{2^{2i-2}p^i-p}}) = \nu(Q_{y,2i}) - (2^{2i-2}p^i - p) = 2^{2i-2}p^i - \frac{1}{2^{2i}} - 2^{2i-2}p^i + p = p - \frac{1}{2^{2i}} = p\beta_{i-1}$.

Therefore for all $i \in \mathbb{N}$ we have $\mu_{i+1}(f) > \mu_i(f) = p - \frac{1}{2^{2i+2}}$. Put $\mu(f) = p > p - \frac{1}{2^{2i+2}} = \mu_i(f)$ for all i .

Suppose that there exists $g \in K[X]$, such that $\deg_x g < \deg_x f = p$ and that $\mu(g) > \mu_i(g)$ for all $i \in \mathbb{N}$. We may assume that g is monic, and that $m = \deg_x g$ is the minimal degree for all the polynomials $\phi \in K[X]$ that satisfy the relation $\mu(\phi) > \mu_i(\phi)$ for all $i \in \mathbb{N}$.

Then there exists i_0 such that for all $i \geq i_0$ in \mathbb{N} , we have :

$$g = Q_i^m + g_{m-1}Q_i^{m-1} + \dots + g_0$$

with $\mu_i(g) = m\beta_i = m\mu_i(Q_i)$.

As $\deg_x g < \deg_x f$, write $f = f_r g^r + f_{r-1}g^{r-1} + \dots + f_0$, with $f_j \in K[X]$ with $\deg_x f_j < \deg_x g$ for all $0 \leq j \leq r$.

For all j , $0 \leq j \leq r$, there exists an $i_{1,j}$ such that for all $i \geq i_{1,j}$,

$$\mu_i(f_j) = \mu_{i+1}(f_j) = \dots = \mu(f_j) = \delta_j.$$

Put $i_2 = \max \{i_0, i_{1,0}, \dots, i_{1,j}, \dots, i_{1,r}\}$, then for all $i \geq i_2$ we have :

$$\mu_i(f) \geq \min_{0 \leq j \leq r} \{\mu_i(f_j g^j)\} = \min_{0 \leq j \leq r} \{\delta_j + jm\beta_i\}.$$

The set $\{\beta_i / i \geq i_2\}$ is infinite, and j and δ_j cannot take but a finite number of values, therefore there exists an $i_3 \geq i_2$, such that for all $i \geq i_3$, $\min_{0 \leq j \leq r} \{\delta_j + jm\beta_i\}$ is attained only once, therefore $\mu_i(f) = \delta_j + jm\beta_i$. On the other hand we have $\mu_i(f) = p\beta_i$ hence considering $i, i' > i_3$ will give : $\mu_i(f) = \delta_j + jm\beta_i = p\beta_i$ and $\mu_{i'}(f) = \delta_j + jm\beta_{i'} = p\beta_{i'}$, therefore subtracting these two equations will give $p = jm$. But p is irreducible and $j \leq r < p$ and $m < p$ which is impossible.

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